Visualizing piecewise-flat manifolds

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November 22, 2010

Background

Definitions

We will call a convex n-polytope in $\mathbb{R}^n$ a **facet**.
The polytopes forming the boundary of a facet will be called **ridges**.
If R is a ridge of a facet F, we will write $R < F$.

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Our data will be

- A finite collection of facets,
- A rule for gluing facets along ridges.

The gluing is specified in the following way:

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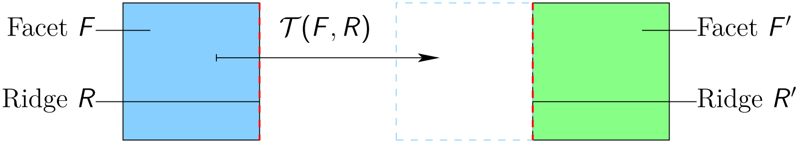
To each pair (F, R) with $R < F$, there is

- A pair $(F', R') = *(F, R)$ with $R' < F'$
- An isometry $T = \mathcal{T}(F, R)$ of $\mathbb{R}^n$



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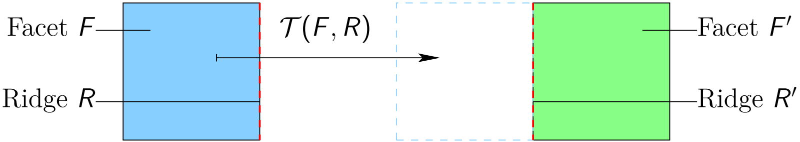

The following should be an equivalence relation:
$(F, p) \sim (F', p')$ if

- $p \in R < F$ and $p' \in R' < F'$, and $*(F, R) = (F', R')$
- $\mathcal{T}(F, R)p = p'$

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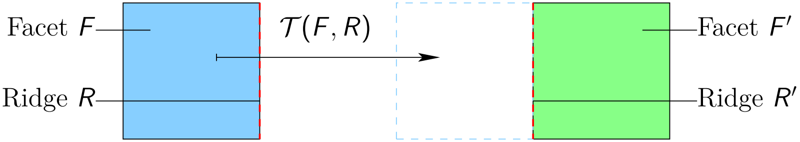

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The space X =(disjoint union of facets)$/\sim$ should be a manifold.

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Points are pairs (F, p) with $p \in F$,
but we may just refer to p if F is understood from context.
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We will call points on the boundary of a ridge **warped points**.
$X \setminus \{$ warped points $\}$ is a flat Riemannian manifold.
This gives a notion of geodesics and an exponential map.

Geodesics

Example on an embedded surface

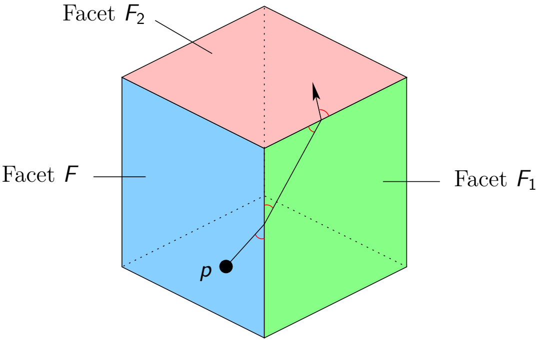

Figure: A view from the embedding.

Geodesics

Example: Step 1

Given $v \in T_pF$, we can compute $\exp_p(v)$ in the following way.

Geodesics

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Given $v \in T_pF$, we can compute $\exp_p(v)$ in the following way.
Let $q = p + v$.

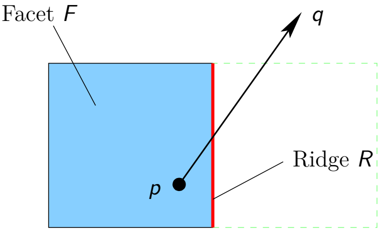

Figure: Since $q \notin F$, apply $T(F,R)$ to p and q.

Geodesics

Example: Step 2

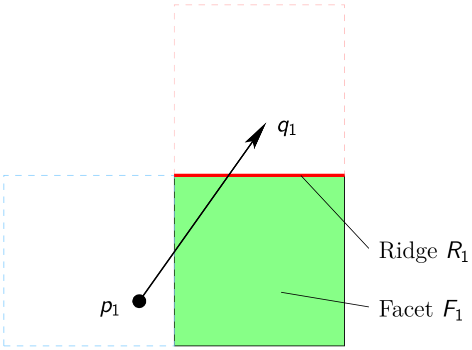

Figure: Since $q_1 \notin F_1$, apply $T(F_1, R_1)$ to p_1 and q_1.

Geodesics

Example: Step 3

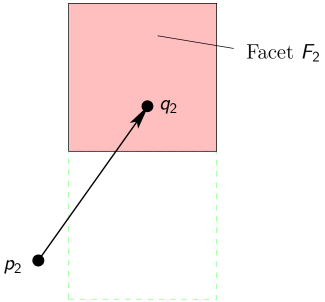

Figure: Since $q_2 \in F_2$, stop; $\exp(v) = (F_2, q_2)$.

Geodesics

Algorithm

Let $v \in T_pF$.
To find $\exp(v)$, let $q = p + v$, and call the following algorithm:

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Algorithm

$\exp(F, p, q)\{$
    $\text{if}(q \in F)\{\ \text{return}(F, q);\ \}$
    $\text{else}\{$
        $R =$ Ridge through which the line segment pq exits F;
        $F' = \mathcal{F}(F, R);$
        $T = \mathcal{T}(F, R);$
        $\text{return}\ \exp(F', Tp, Tq);$
    $\}$
$\}$

Geodesics

Some useful remarks

We saw that for $v \in T_pF$, $\exp_p(v) = T(p + v)$ for some isometry T.
If we trace the path $\exp_p(tv)$ as t goes from 0 to 1,
it would pass through some sequence of facets and ridges.
This sequence determines the transformation T.

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We can "parallel transport" vectors along $\exp_p(tv)$ from p to $\exp_p(v)$:
If $\exp_p(v) = T(p+v)$, then for $w \in T_pF$, define

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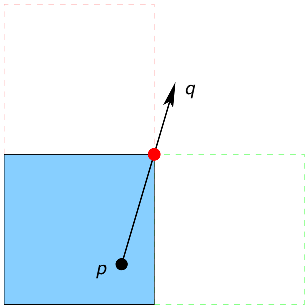
Now we can move at a consistent velocity in our space.
If $w \in T_pF$ is the velocity, and Δt seconds have passed,

- Set the new position to $\exp_p(w \cdot \Delta t)$
- Set the new velocity to $P_{w\cdot\Delta t}(w)$.

Geodesics

Passing through warped points?

How does the algorithm work in the following case?



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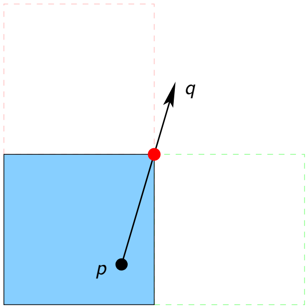
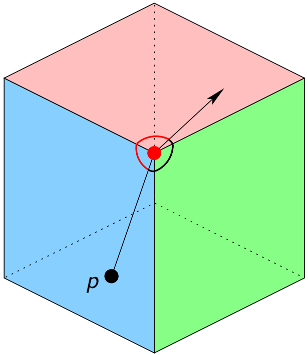


We usually say that v is not in the domain of $\exp$.

Geodesics

Passing through warped points?

An alternative (in 2-dimensions) is the “straightest geodesic”.



The path is continued in such a way that

$$(\text{sum of red angles}) = (\text{sum of black angles})$$

This can be computed by “walking” around the vertex.

Geodesics

First step to visualization

Now we can figure out what we would “see” from a position p.
Let’s see what kind of phenomena we can expect to occur.

Geodesics

First step to visualization

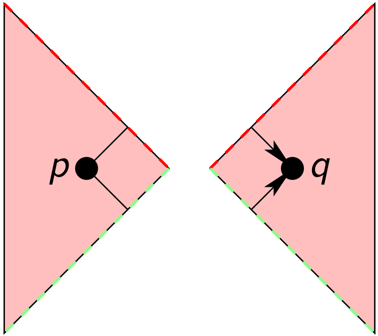


Figure: $(\kappa > 0)$ I see two copies of q from p.

Geodesics

First step to visualization

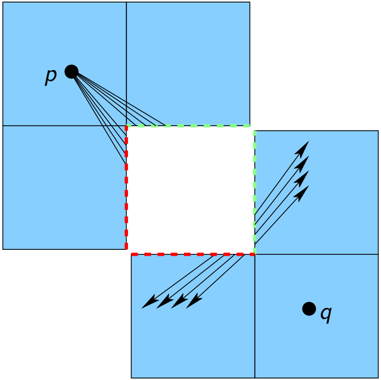


Figure: ($\kappa < 0$) I cannot see q from p.

Visualization

What are we trying to do?

Fix a facet F_0 and a point $p \in F_0$.
We will call T_pF_0 the **visual field** for an observer at p.
If the observer looks in a direction and distance $v \in T_pF_0$, what (s)he sees is whatever is at the point $\exp(v)$.

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Fix a facet F_0 and a point $p \in F_0$.
We will call T_pF_0 the **visual field** for an observer at p.
If the observer looks in a direction and distance $v \in T_pF_0$, what (s)he sees is whatever is at the point $\exp(v)$.

So, we want to identify $T_pF_0 \simeq \mathbb{R}^n$ and "populate" the visual field by putting at v whatever is in the space at $\exp(v)$.
We can't do this point-by-point, but we will make some observations allowing us to do this efficiently.

Visualization

Observation 1

For v in the domain of $\exp$, consider the path $\exp(tv)$ for $t \in [0, 1]$.
The path passes through some sequence of facets and ridges.

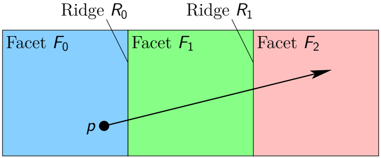

Figure: Here, the sequence is $(F_0, R_0, F_1, R_1, F_2)$.

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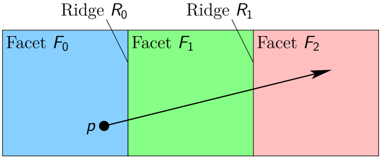

Figure: Here, the sequence is $(F_0, R_0, F_1, R_1, F_2)$.

All v in the domain of $\exp$ have such a sequence $\mathcal{S}(v)$.
The domain of $\exp$ can be partitioned into sets $\mathcal{D}(S) = \{v | \mathcal{S}(v) = S\}$.

Visualization

Observation 2

On each of the sets $\mathcal{D}(S)$, $\exp$ has a simple form:
For $S = (F_0, R_0, \ldots, F_k, R_k, F)$ and $v \in \mathcal{D}(S)$,

$$\exp(v) = (\mathcal{T}(F_k, R_k) \circ \cdots \circ \mathcal{T}(F_0, R_0))\,(p + v).$$

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Let $\mathcal{E}(S) = \exp(\mathcal{D}(S)) \subset F$.
We can draw the whole chunk $\mathcal{D}(S)$ of the visual field at once,
by applying the inverse of the above isometry to $\mathcal{E}(S)$
(and any objects within).

The collection of sequences forms a tree, Γ:

- The root of the tree is (F_0).
- The children of $(\ldots, F)$ are the sequences $(\ldots, F, R_i, F_i)$, where $R_i < F$ and $F_i = \mathcal{F}(F, R_i)$.

Visualization

Observation 3

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If we can find $\mathcal{E}(S')$ for children S' of S, where $\mathcal{E}(S)$ is known, then by traversing Γ we can draw the entire visual field.

Frustums
Definition

The fourth observation is that the sets $\mathcal{E}(S)$ have a particularly nice form. To state the observation, we make the following definition:

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Definition (Frustum)

A **frustum** is a subset $V \subset \mathbb{R}^n$ such that there exists

- a **source point** $p \in \mathbb{R}^n$ and
- a convex $(n-1)$-polytope $Q \not\ni p$,

such that $V = \{p + k(q-p) | q \in Q, k \geq 0\}$.

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It will be convenient to consider $\emptyset$ and $\mathbb{R}^n$ to be a frustums (the **empty frustum** and **full frustum**, respectively).

Frustums

Examples

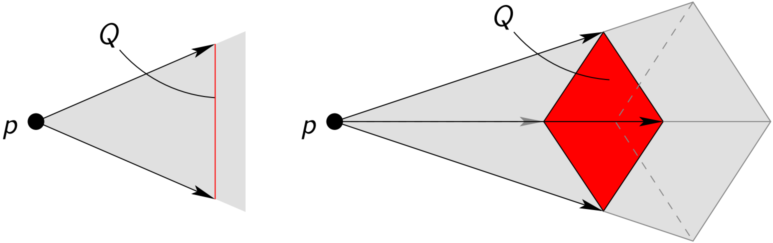
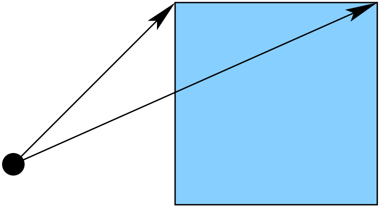


Figure: Examples of frustums.

Frustums

Creation

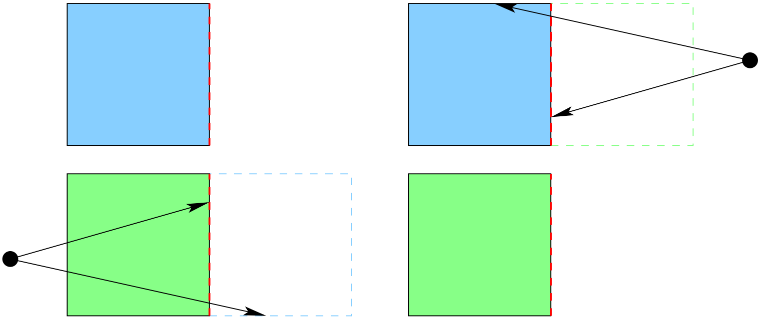
A frustum can be created from a point and a ridge.



Frustums

Transformation

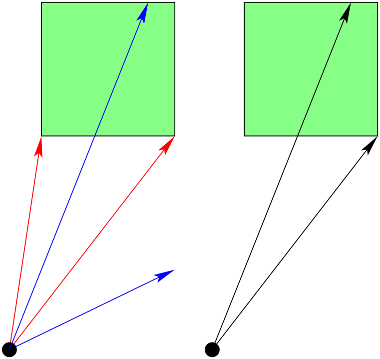
A frustum can be transformed along a ridge.



Frustums

Intersection

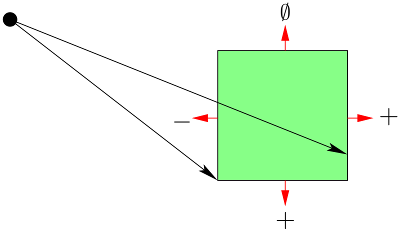
Frustums can be intersected to get a new frustum (which may be empty).



Frustums

Intersection

A frustum V with source point p intersects a ridge R **positively** if
$\exists v \in V \cap R$ such that $(v - p) \cdot n > 0$,
where n is the outward pointing normal to R.



Visualization

Observation 4

Claim: For every sequence $S = (\ldots, F)$, there is a frustum $\mathcal{V}(S)$ such that $\mathcal{E}(S) = \mathcal{V}(S) \cap F$.

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To prove this (and also complete the algorithm to draw the visual field)

- We show this is true for the root of Γ
- We compute $\mathcal{V}(S')$, where S' is a child of S and $\mathcal{V}(S)$ is known.

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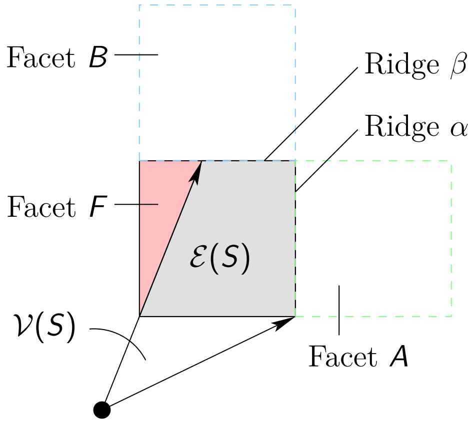
- We compute $\mathcal{V}(S')$, where S' is a child of S and $\mathcal{V}(S)$ is known.

The claim is trivial for the root (F_0) of Γ, since $\mathcal{E}(F_0) = \mathbb{R}^n \cap F_0$.
For the inductive step, we will do an example.

Visualization

The inductive step

Data for sequence $S = (\dots, F)$

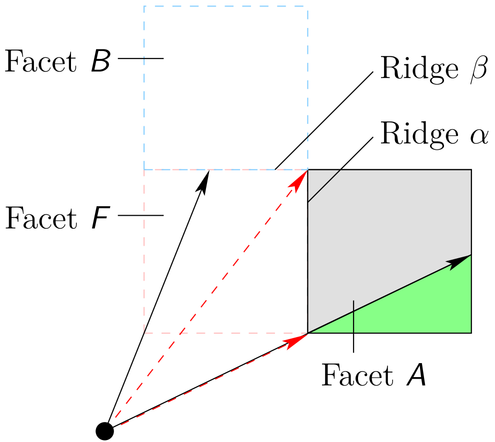


The frustum intersects ridges α and β positively.

Visualization

The inductive step

Data for child node $S' = (\ldots, F, \alpha, A)$

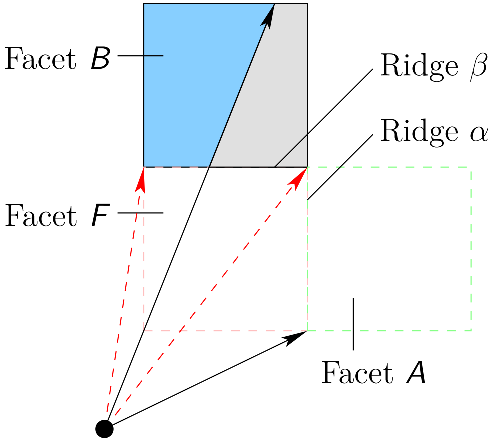


$\mathcal{V}(S') = (\text{red frustum}) \cap (\text{black frustum})$

Visualization

The inductive step

Data for child node $S'' = (\ldots, F, \beta, B)$



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Visualization

The algorithm

Let $S = (\ldots, F)$ be a sequence for which $\mathcal{V}(S)$ is known.
Let $S' = (\ldots, F, R, F')$ be a child of S.
The following returns $\mathcal{V}(S')$:

Visualization

The algorithm

Let $S = (\dots, F)$ be a sequence for which $\mathcal{V}(S)$ is known.
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The following returns $\mathcal{V}(S')$:

Algorithm

if($\mathcal{V}(S)$ intersects R positively){
 p = Source point of $\mathcal{V}(S)$;
 V_R = Frustum generated from p and R;
 return $\mathcal{T}(F, R)(V_R \cap \mathcal{V}(S))$;
}
else{ return $\emptyset$; }