Desingularizing the Boundary of the Moduli Space of Genus One Stable Quotients

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Background

Moduli Spaces

What is a moduli space?

Example: Smooth plane conics.

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$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz = 0$$

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$\mathbb{P}^5$ is a **moduli space** of plane conics:

$$\left\{\text{Points of } \mathbb{P}^5\right\} \xleftrightarrow{1-1} \{\text{Plane conics}\}$$

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This is a degree 3 polynomial in the coordinates of $\mathbb{P}^5$.
It defines a hypersurface $\Delta \subset \mathbb{P}^5$.

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What is the boundary like?

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$$(\alpha x + \beta y + \gamma z)^2 = 0$$

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- Blowing up along Δ_{double} will resolve the singularities in the boundary.

The **moduli space of genus one stable quotients** is a nonsingular compactification of the moduli space of maps from smooth genus 1 curves into projective space.

(By curve we mean a 1-dimensional complex projective variety)

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Maps and Quot Schemes

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- Define a map by:
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Degree d maps correspond to the case $\deg(S^\vee) = d$.

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Maps and Quot Schemes

The moduli space of maps $C \to \mathbb{P}^{n-1}$ sits inside of a **Quot scheme**.

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$$\left\{ \begin{array}{c} \text{Maps} \\ T \to \mathrm{Quot}^{r,d}_{E/\mathcal{C}_B/B} \end{array} \right\} \overset{1-1}{\longleftrightarrow} \left\{ \begin{array}{c} \text{Families of quotients of } E \\ \text{parameterized by } T \\ \text{with rank } r \text{ and relative degree } d \end{array} \right\}$$

Meaning:

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Meaning:

(i) A map $f : T \to B$,

(ii) A quotient of $E_T = \overline{f}^* E$ on $\mathcal{C}_T = T \times_B \mathcal{C}_B$, flat over T:

$$\begin{array}{ccccc} [E_T \twoheadrightarrow Q] & \rightsquigarrow & \mathcal{C}_T & \xrightarrow{\overline{f}} & \mathcal{C}_B \\ & & \downarrow & & \downarrow \\ & & T & \xrightarrow{f} & B \end{array}$$

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Consider $Q_d = \mathrm{Quot}^{n-1,d}_{\mathcal{O}_{\mathbb{P}^1}^{\oplus n}/\mathbb{P}^1/\mathbb{C}}$.

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- Try to define $\mathbb{P}^1 \to \mathbb{P}^{n-1} : p \mapsto [s_1(p), \cdots, s_n(p)]$ like before.
- Now the s_i can all vanish at the same point (*rational* maps). The degree of $\tau(Q)$ at p = common order of vanishing of $s_i(p)$.

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- The boundary has a filtration

$$Z_{d,0} \hookrightarrow Z_{d,1} \hookrightarrow \cdots \hookrightarrow Z_{d,d-1} \hookrightarrow Q_d$$

where $Z_{d,k} = \{\text{Quotients w/ degree of torsion} \geq d - k\}$

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- On $\mathbb{P}^1_{Q_d}$, the locus where all s_i vanish has codimension n. The image in Q_d is $Z_{d,d-1}$, so the boundary has codimension $n-1$.

Yijun Shao carried out a blow-up procedure on Q_d yielding a boundary which is a simple normal crossings divisor.

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(Picture: A collection of smooth curves stuck together at some points)

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A **semi-stable** genus 1 curve is smooth or a cycle of $\mathbb{P}^1$'s.

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satisfying:

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- **Stable** means (in this setting):
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$\mathcal{Q}_d$ (resp. $\widetilde{\mathcal{Q}}_d$) = moduli space of stable (resp. quasi-stable) quotients.

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  (It can be shown that these are smooth)
- For a fixed family of curves $U \to \mathcal{M}_1$, there is an open embedding

$$U \times_{\mathcal{M}_1} \widetilde{\mathcal{Q}}_d \hookrightarrow \mathrm{Quot}^{n-1,d}_{\mathcal{O}^{\oplus n}_{\mathcal{C}_U}/\mathcal{C}_U/U}$$

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Theorem (Marian, Oprea, Pandharipande): $\mathcal{Q}_d$ is a nonsingular, irreducible, separated, proper Deligne-Mumford stack of finite type over $\mathbb{C}$, of dimension nd.

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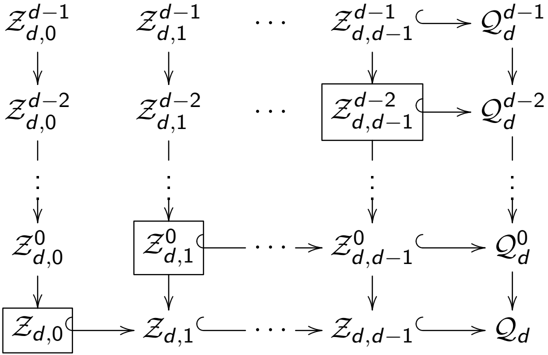
We will adapt the blow-up process for Q_d ($g = 0$) to $\mathcal{Q}_d$ ($g = 1$).
Goal: Show the resulting boundary = divisor with simple normal crossings.

Background

Blowing up

Blow up each row along the space indicated by a box.



Theorem: $\mathcal{Z}_{d,0}^{d-1}, \dots, \mathcal{Z}_{d,d-1}^{d-1}$ are nonsingular, codimension 1, and intersect transversally in $\mathcal{Q}_d^{d-1}$.

The Setup

Defining the Scheme Structure

Consider the universal sequence on $\mathcal{C}_{\mathcal{Q}_d}$:

$$0 \to \mathcal{S} \to \mathcal{O}_{\mathcal{C}_{\mathcal{Q}_d}}^{\oplus n} \to \mathcal{Q} \to 0$$

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Dualize, twist, and push down to $\mathcal{Q}_d$:

$$\pi_* {\mathcal{O}_{\mathcal{C}_{\mathcal{Q}_d}}^{\oplus n}}^\vee(m) \xrightarrow{\rho_m} \pi_* \mathcal{S}^\vee(m)$$

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For $m \gg 0$, this is a map of bundles on $\mathcal{Q}_d$.

The Setup

Defining the Scheme Structure

For $q = (C, \mathcal{O}_C^{\oplus n} \twoheadrightarrow Q) \in \mathcal{Q}_d$,

$$\operatorname{rank} \rho_m|_q = mD + d - \deg \tau(Q)$$

(D is the degree of the ample line bundle on $\mathcal{C}_{\mathcal{Q}_d}$)

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Hence:

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Define $\mathcal{Z}_{d,k}$ to be the vanishing of $\bigwedge^{mD+k+1} \rho_m$.

The Setup

Preparing for Induction: Factorizations

To use inductive arguments, we want to relate the degree d procedure to the degree k procedure for $k < d$.

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(Stability condition: $\omega_C \otimes (K^\vee)^{\otimes \epsilon}$ is ample for all $0 < \epsilon \in \mathbb{Q}$)

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It can be shown that Q fits into the short exact sequence:

$$0 \to T \to Q \to F \to 0.$$

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Define $\mathcal{Q}_{d,k}$ to be the moduli space of these pairs ("factorizations").

There are maps:

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What if we require the degree k quotient to be stable?

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Preparing for Induction: Factorizations

Define $\mathcal{Q}_{d,k}$ to be the moduli space of these pairs ("factorizations").

There are maps:

$$\widetilde{\mathcal{Q}}_k \xleftarrow[\text{forget}]{\theta} \mathcal{Q}_{d,k} \xrightarrow[\text{compose}]{\phi} \mathcal{Q}_d$$

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Conclusion: To use induction we will have to blow up $\widetilde{\mathcal{Q}}_d$.

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Working smooth-locally

Problem: What do we blow up on $\widetilde{\mathcal{Q}}_d$?

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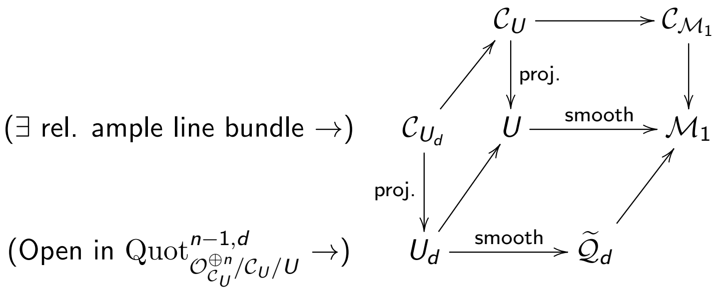
Solution: Cover $\mathcal{M}_1$ by smooth $U \to \mathcal{M}_1$ with $\mathcal{C}_U \to U$ projective:



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Define $V_{d,k}$ to be the vanishing of $\bigwedge^{mD+k+1} \rho_m$.

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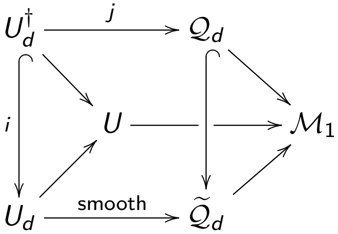

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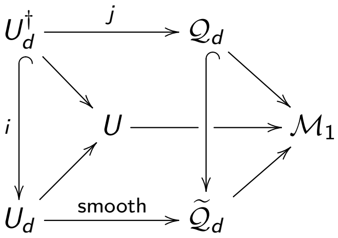

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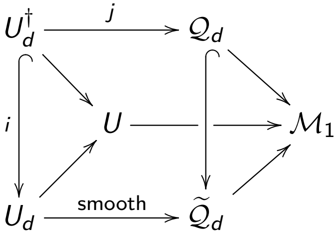

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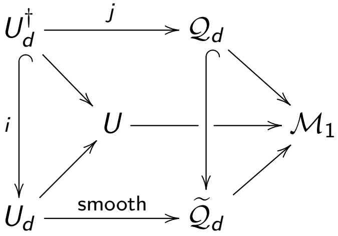

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- i is an open embedding and j is smooth,
- $i^{-1}(V_{d,k}) = j^{-1}(\mathcal{Z}_{d,k})$ ($\leftarrow$ Different bundles used for twisting)

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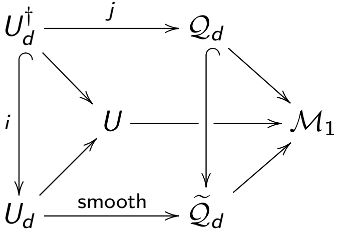

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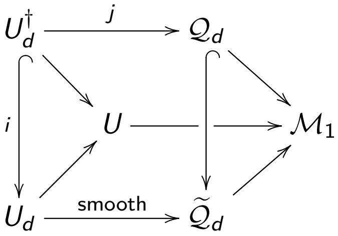

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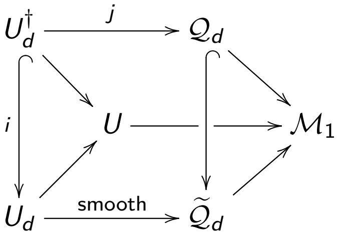

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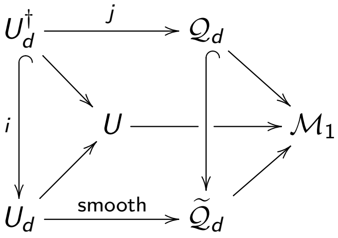

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- $\widetilde{j}^{-1}(\mathcal{Z}_{d,k}^r) = (V_{d,k}^r)^\dagger$.
- If **P** is smooth-local, $\{V_{d,k}^{d-1}\}$ satisfy **P** $\implies$ $\{\mathcal{Z}_{d,k}^{d-1}\}$ satisfy **P**.

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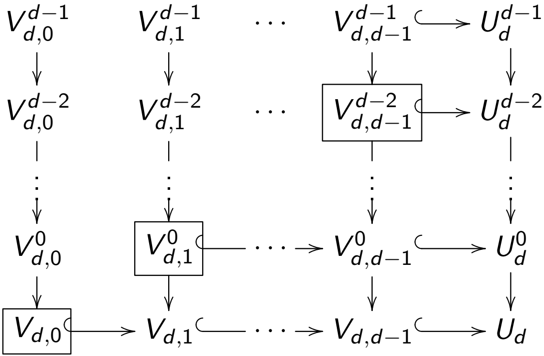
The blow-up process on U_d:



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$$\begin{aligned} \mathring{V}_{d,k} &\simeq \mathring{U}_{d,k} \ (\leftarrow U_{d,k} \times_{U_k} \mathring{U}_k) \\ V_{d,k}^{k-1} &\simeq U_{d,k} \times_{U_k} U_k^{k-1} \end{aligned}$$

More Details

Space of Collineations

We can embed U_d into a space of collineations.
Given bundles E, F on X, define $\mathbb{S}(E, F) = \mathbb{P}(\mathcal{H}om(E, F))$.

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Results of Vainsencher $\implies$ $U_d^k \left(= \mathrm{Bl}_{V_{d,k}^{k-1}} U_d^{k-1}\right) = \mathrm{Bl}_{b^{-1}(V_{d,k})} U_d^{k-1}$

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Using the result of Vainsencher, the closure of the image is U_d^k.

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- Is this useful for $g > 1$? The moduli of stable quotients is singular.

References

- A. Marian, D. Oprea, R. Pandharipande, *The moduli space of stable quotients*.
- Y. Shao, *A compactification of the space of parameterized rational curves in Grassmannians*.
- I. Vainsencher, *Complete collineations and blowing up determinantal ideals*.