Moduli Spaces and Enumerative Geometry

Thomas D. Maienschein

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(1) Projective space and Bézout's theorem,
(2) The moduli space of plane conics,
(3) The moduli space of stable quotients (sort of).

Section 1

Projective Space and Bézout's Theorem

Projective Space and Bézout's Theorem

Basic Observations

Let's look at some zero loci of polynomials in $\mathbb{R}^2$.

Projective Space and Bézout's Theorem

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Denote by $Z(f)$ the zero locus of a polynomial f.

Projective Space and Bézout's Theorem

Basic Observations

Polynomial degrees 1 and 1; intersection points: $\boxed{1}$

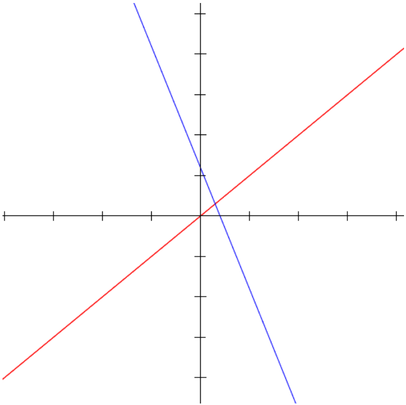

Figure: $Z(y - x)$, $Z(y + 3x - 1)$

Projective Space and Bézout's Theorem

Basic Observations

Polynomial degrees 1 and 2; intersection points: $\boxed{2}$

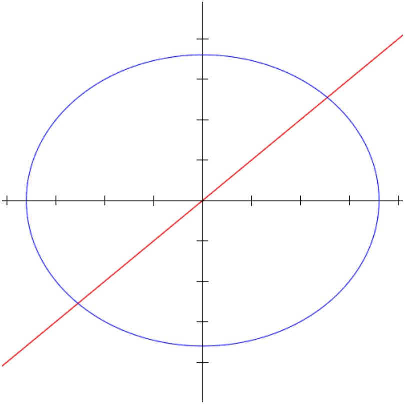

Figure: $Z(y - x),\ \ Z(x^2 + y^2 - 9)$

Projective Space and Bézout's Theorem

Basic Observations

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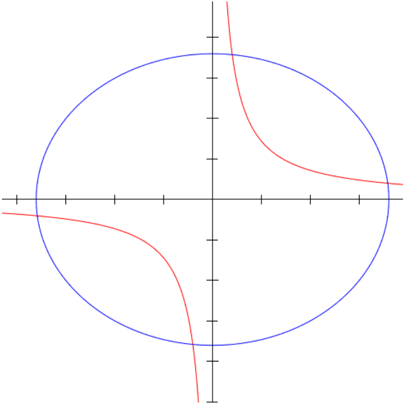

Figure: $Z(xy-1)$, $Z(x^2+y^2-9)$

Projective Space and Bézout's Theorem

Basic Observations

Polynomial degrees 2 and 3; intersection points: $\boxed{6}$

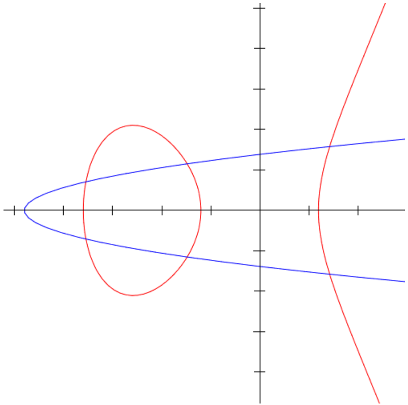

Figure: $Z(y^2 - x^3 - 3x^2 + x + 3)$, $Z(x - 3y^2 + 4)$

Question

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$$\#Z(f) \cap Z(g) = (\deg f)(\deg g)?$$

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(iv) The intersection "occurs at infinity" ($y = 0$ and $y = 1$).

We can't fix infinite intersections, so this will be an exception.

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Allowing $\mathbb{C}$ and counting multiplicity should be familiar.

The fundamental theorem of algebra says

$$\#Z\big(y - f(x)\big) \cap Z(y) = \deg f.$$

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Let's see how to handle "intersections at infinity" using projective space.

Projective Space and Bézout's Theorem

Basic Observations

As the slope of the line increases, an intersection point goes to infinity.

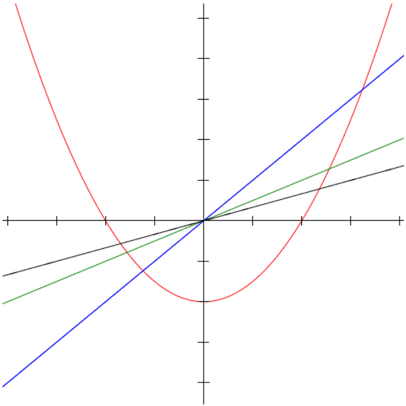

Figure: $Z(y - x^2)$, $Z(y - ax)$ with a increasing.

Projective Space and Bézout's Theorem

Basic Observations

The intersection is missing a point: Back

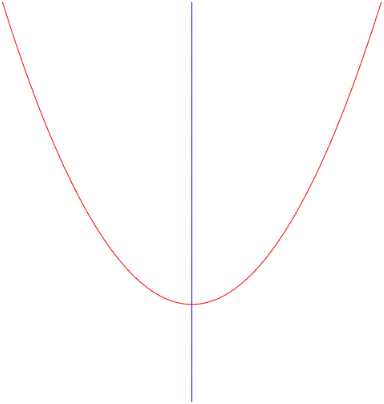

Figure: $Z(y - x^2), \;\; Z(x)$

Projective Space and Bézout's Theorem

Projective Space

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Denote a point in $\mathbb{P}^n$ by $[x_0 : x_1 : \cdots : x_n]$, at least one $x_i \neq 0$.
This represents the line $\left\{(\lambda x_0, \lambda x_1, \ldots, \lambda x_n) \,|\, \lambda \in \mathbb{R}\right\}$.

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This represents the line $\{(\lambda x_0, \lambda x_1, \ldots, \lambda x_n) \mid \lambda \in \mathbb{R}\}$.

Note that

$$[x_0 : x_1 : \cdots : x_n] = [\lambda x_0 : \lambda x_1 : \cdots : \lambda x_n]$$

for any $\lambda \neq 0$.

Consider $U = \{[x : y : z] \mid z \neq 0\} \subset \mathbb{P}^2$.
Any such point can be represented uniquely as $[X : Y : 1]$.
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It consists of the lines in the plane $z = 0$.
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Projective Space and Bézout's Theorem

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In general, $\mathbb{P}^n$ looks like $\mathbb{R}^n$, with an extra $\mathbb{P}^{n-1}$ "at infinity".

Projective Space and Bézout's Theorem

Homogenization

A polynomial is **homogeneous** if all its terms have the same degree.
We can **homogenize** any polynomial $f(x, y)$.

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$$\begin{aligned} f &= x^3 - 2xy + y - 5 \\ f_h &= x^3 - 2xyz + yz^2 - 5z^3 \end{aligned}$$

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(ii) $f_h(x, y, 1) = 0$ exactly when $f(x, y) = 0$.
So $Z(f)$ is the intersection of $Z(f_h)$ with $\mathbb{R}^2 \subset \mathbb{P}^2$.

Projective Space and Bézout's Theorem

Homogenization

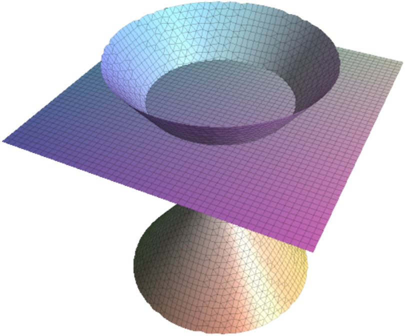


Figure: $f = x^2 + y^2 - 1, \;\; f_h = x^2 + y^2 - z^2$

Projective Space and Bézout's Theorem

Homogenization

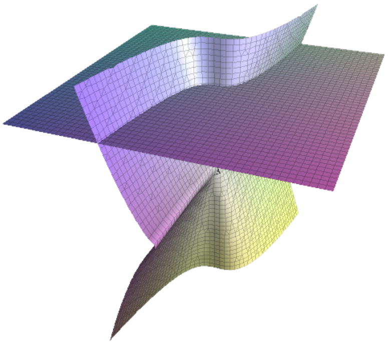


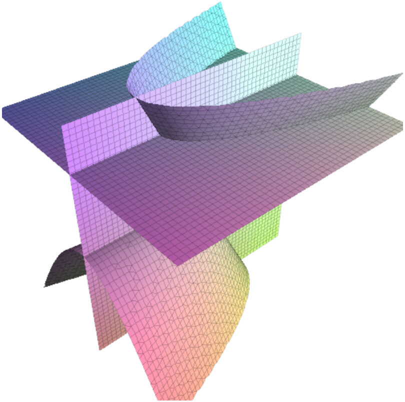
Figure: $f = y - x^3, \;\; f_h = yz^2 - x^3$

Projective Space and Bézout's Theorem

Homogenization

Line and conic that intersect "at infinity": ▸ Original

$$y = x^2 - 1 \qquad (f_h = yz - x^2 + z^2)$$
$$x = 0 \qquad (g_h = x)$$



Projective Space and Bézout's Theorem

Homogenization

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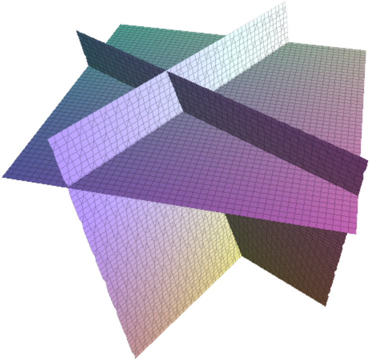
- At "infinity": $[0 : 1 : 0]$.

Projective Space and Bézout's Theorem

Homogenization

Lines that intersect in $\mathbb{R}^2$:

$$x + 2y + 2 = 0 \qquad (f_h = x + 2y + 2z)$$
$$-2x + y - 1 = 0 \qquad (g_h = -2x + y - z)$$

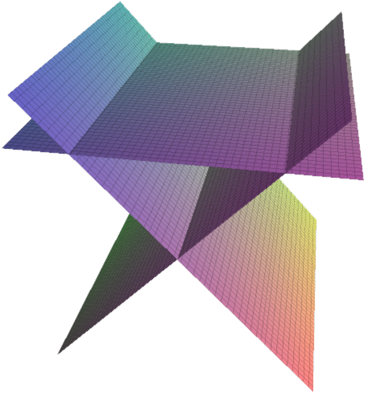


Projective Space and Bézout's Theorem

Homogenization

Lines that intersect "at infinity":

$$x = 3 \qquad (f_h = x - 3z)$$
$$x = -3 \qquad (g_h = x + 3z)$$

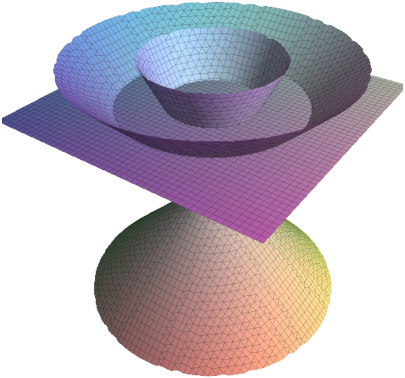


Projective Space and Bézout's Theorem

Homogenization

We were using $\mathbb{RP}^2$, but of course we need $\mathbb{CP}^2$:

$$x^2 + y^2 = 1 \qquad (f_h = x^2 + y^2 - z^2)$$
$$x^2 + y^2 = 4 \qquad (g_h = x^2 + y^2 - 4z^2)$$



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A variety $X \subset \mathbb{P}^n$ has a **degree**.

- The hypersurface $Z(f)$ has degree $\deg f$.
- If $X_1, \ldots, X_n$ are hypersurfaces with 0-dimensional intersection,

$$\# \bigcap X_i = \prod \deg X_i$$

This is **Bézout's Theorem**.

Section 2

Classical Example: Plane Conics

Classical Example: Plane Conics

Moduli space of conics

A **plane conic** is a curve in $\mathbb{P}^2$ defined by a degree 2 polynomial:

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz = 0$$

Classical Example: Plane Conics

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We can regard this as a point $[A : B : C : D : E : F] \in \mathbb{P}^5$.

$\mathbb{P}^5$ is a **moduli space** of plane conics.

Classical Example: Plane Conics

Moduli space of conics

Example

Conic: $xy = 1$. Homogenize: $xy - z^2 = 0$.

Classical Example: Plane Conics

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Conic: $xy = 1$. Homogenize: $xy - z^2 = 0$.

This lives at the point $[0 : 0 : -1 : 1 : 0 : 0]$.

Example

Conic: $xy = 0$. This is a pair of lines.

Classical Example: Plane Conics

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Classical Example: Plane Conics

Moduli space of conics

A conic is **singular** if it is a pair of lines ($xy = 0$).

This happens exactly when:

$$\begin{vmatrix} 2A & D & E \\ D & 2B & F \\ E & F & 2C \end{vmatrix} = 0.^*$$

This cuts out a degree 3 hypersurface $\Delta \subset \mathbb{P}^5$.

$^*4ABC + DEF - AF^2 - BE^2 - CD^2 = 0$

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$\mathbb{P}^5$ is a **compactification** of the moduli space of *smooth* conics,
Δ is the **boundary** of the moduli space.

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Classical Example: Plane Conics

Moduli space of conics

Inside of Δ live the **double lines**:

$$(\alpha x + \beta y + \gamma z)^2 = 0$$

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Moduli space of conics

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The locus Δ_{double} is exactly the singular locus of Δ.

Δ_{double} is the image of the Veronese embedding:

$$\mathbb{P}^2 \hookrightarrow \mathbb{P}^5 : [\alpha : \beta : \gamma] \mapsto [\alpha^2 : \beta^2 : \gamma^2 : 2\alpha\beta : 2\alpha\gamma : 2\beta\gamma]$$

Classical Example: Plane Conics

Passing through points and tangent to lines

Fix $P \in \mathbb{P}^2$. Let $Z_P = \{\text{conics passing through } P\} \subset \mathbb{P}^5$.

Classical Example: Plane Conics

Passing through points and tangent to lines

Fix $P \in \mathbb{P}^2$. Let $Z_P = \{\text{conics passing through } P\} \subset \mathbb{P}^5$.

This is a hyperplane in $\mathbb{P}^5$:

Write $P = [x_0 : y_0 : z_0]$. Find $[A : B : C : D : E : F]$ so that

$$(x_0^2)A + (y_0^2)B + (z_0^2)C + (x_0 y_0)D + (x_0 z_0)E + (y_0 z_0)F = 0$$

This is just a linear equation in A, B, C, D, E, F.

Classical Example: Plane Conics

Passing through points and tangent to lines

Question

Fix 5 points $P_1, \ldots, P_5 \in \mathbb{P}^2$.

How many conics pass through all of $P_1, \ldots, P_5$?

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Generically, the hyperplanes Z_{P_i} will intersect transversally.

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If no 3 points are collinear, nothing in Δ can pass through all 5 points.

Classical Example: Plane Conics

Passing through points and tangent to lines

Fix a line $L \subset \mathbb{P}^2$. Let $Z_L = \{\text{conics tangent to } L\} \subset \mathbb{P}^5$.[†]

[†] C is tangent to L if $\#C \cap L = 1$ (with multiplicity 2).

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Fix a line $L \subset \mathbb{P}^2$. Let $Z_L = \{\text{conics tangent to } L\} \subset \mathbb{P}^5$.[†]

This is a degree 2 hypersurface in $\mathbb{P}^5$ (next slide).

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Since the Z_P's and Z_L's are all hypersurfaces in $\mathbb{P}^5$,
intersecting 5 of them should give finitely many points to count.

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Classical Example: Plane Conics

Passing through points and tangent to lines

Suppose L is the line $x = 0$.

Classical Example: Plane Conics

Passing through points and tangent to lines

Suppose L is the line $x = 0$.

Find $[A : B : C : D : E : F]$ so that the system

$$\begin{cases} x = 0 \\ Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz = 0 \end{cases}$$

has < 2 solutions.

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Look in the chart $z = 1$. Set $x = 0$.

$$By^2 + Fy + C = 0$$

Classical Example: Plane Conics

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$$By^2 + Fy + C = 0$$

This has < 2 solutions when its discriminant is zero:

$$F^2 - 4BC = 0$$

Classical Example: Plane Conics

Passing through points and tangent to lines

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Fix 4 points $P_1, \ldots, P_4 \in \mathbb{P}^2$ and a line $L \subset \mathbb{P}^2$.

How many conics pass through $P_1, \ldots, P_4$ and are tangent to L?

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Bézout:

$$\# Z_{P_1} \cap Z_{P_2} \cap Z_{P_3} \cap Z_{P_4} \cap Z_L = 1 \cdot 1 \cdot 1 \cdot 1 \cdot 2 = \boxed{2}$$

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These will be smooth: $\bigcap Z_{P_i}$ will generally contain 3 singular conics, but they won't be tangent to a general line.

Classical Example: Plane Conics

Passing through points and tangent to lines

Question

Fix 5 lines $L_1, \ldots, L_5 \subset \mathbb{P}^2$.

How many conics are tangent to all of $L_1, \ldots, L_5$?

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This is wrong! The problem is that each Z_{L_i} contains *all* of Δ_{double}. Bézout doesn't apply because $\bigcap Z_{L_i}$ isn't finite.

Classical Example: Plane Conics

Passing through points and tangent to lines

Question

Fix 5 lines $L_1, \ldots, L_5 \subset \mathbb{P}^2$.

How many conics are tangent to all of $L_1, \ldots, L_5$?

Bézout:

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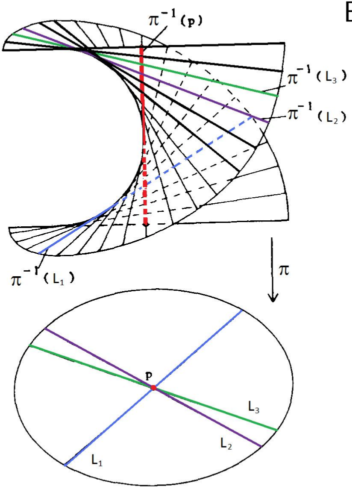
This is wrong! The problem is that each Z_{L_i} contains *all* of Δ_{double}. Bézout doesn't apply because $\bigcap Z_{L_i}$ isn't finite.

One fix is to **blow up** Δ_{double}. This will separate the Z_{L_i}.

Classical Example: Plane Conics

Passing through points and tangent to lines



Blowing up $\mathbb{C}^2$ at P ($\mathrm{Bl}_P\mathbb{C}^2$).

Classical Example: Plane Conics

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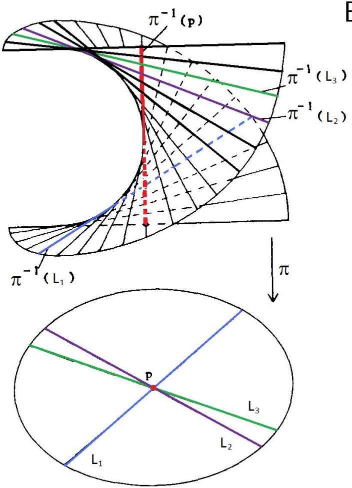

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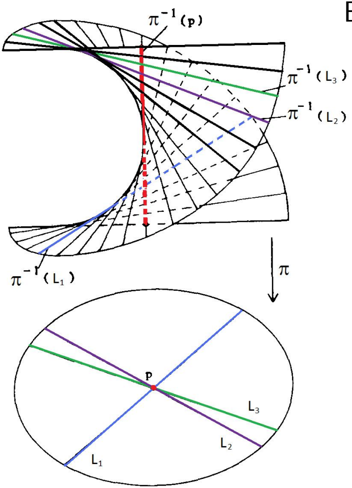

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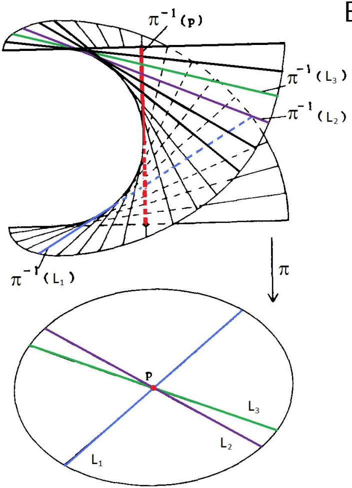

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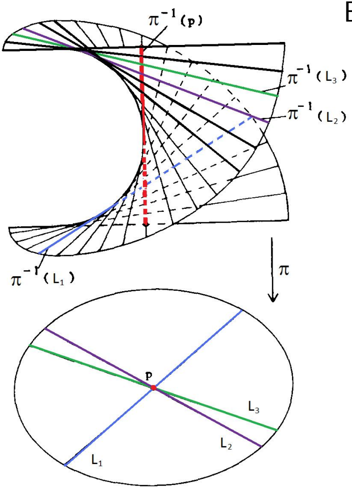

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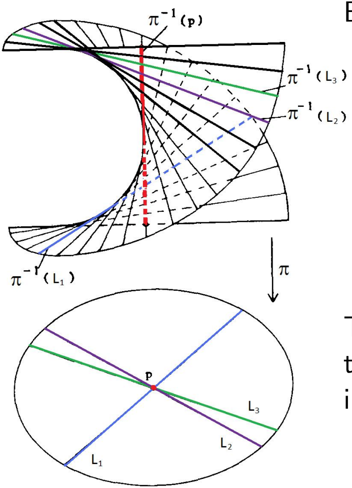

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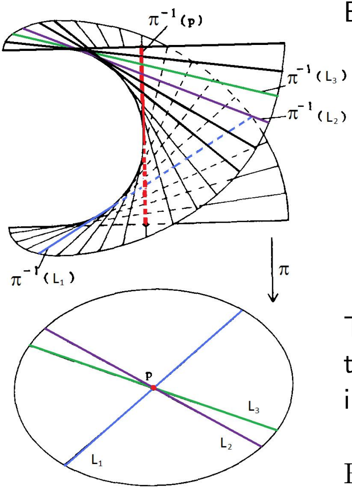

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$\mathrm{Bl}_{\Delta_{\mathrm{double}}}\mathbb{P}^5$ = **complete conics**.

Section 3

Modern Example: Maps to Projective Space

Modern Example: Maps to Projective Space

Maps to Projective Space

A Riemann surface C is a compact surface that locally looks like $\mathbb{C}$.

We want to build a **moduli space** Q of pairs (C, f) where

- C is a Riemann surface (of some fixed genus),
- f is a map $C \to \mathbb{P}^n$.

Modern Example: Maps to Projective Space

Maps to Projective Space

We want to build a **moduli space** Q of pairs (C, f) where

- C is a Riemann surface (of some fixed genus),
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That is, points of Q exactly correspond to such pairs (C, f).

Modern Example: Maps to Projective Space

Maps to Projective Space

How can we get a map $C \to \mathbb{P}^n$?

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- We need to use **global sections of line bundles** instead.
- Line bundles have a **degree**.
- If $\deg \mathcal{L} = d$, global sections vanish at d points (multiplicity).

Modern Example: Maps to Projective Space

Maps to Projective Space

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If $s_0(p) = \cdots = s_n(p) = 0$, our map is not defined at p.

A **rational map** is defined everywhere except finitely many points.

Degree of our rational map = degree of the line bundle the s_i come from.

Modern Example: Maps to Projective Space

The moduli space of maps

Using fancy but standard techniques,
We can build a compact **moduli space** Q:

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The **boundary** consists of (C, f) where f is not defined everywhere.

To recap:

Modern Example: Maps to Projective Space

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We built a moduli space of “plane conics”.
It was compact, at the cost of including singular conics.

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To recap:

We built a moduli space of "plane conics".
It was compact, at the cost of including singular conics.

We can build a moduli space of "maps" $C \to \mathbb{P}^n$.
It is compact, at the cost of including rational maps.

Modern Example: Maps to Projective Space

The Boundary

Let's investigate the **boundary** of Q.

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For each $p \in C$, add the minimum of

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Define $Z_k \subset Q$ to be

$$\Big\{ (C, [s_0 : \cdots : s_n]) \,|\, s_i \text{ vanish simultaneously to order at } \textit{least } k \Big\}.$$

Modern Example: Maps to Projective Space

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For plane conics, the boundary looked like:

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Here the boundary looks like:

$$Z_d \subset Z_{d-1} \subset \cdots \subset Z_1 \subset Q$$

Ideal boundary for intersection theory ($\star$):

Modern Example: Maps to Projective Space

Blowing Up

Ideal boundary for intersection theory $(\star)$:

- Made of some codimension 1 pieces,

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In the plane conic example, Δ was singular.

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Blowing Up

Ideal boundary for intersection theory ($\star$):

- Made of some codimension 1 pieces,
- Each piece is non-singular,
- The pieces are not tangent to one another.

In the plane conic example, Δ was singular.
When we blow up $\mathbb{P}^5$ along Δ_{double}, the new boundary satisfies ($\star$).

Modern Example: Maps to Projective Space

Blowing Up

The boundary Z_1 of Q is singular and codimension $n-1$.

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The boundary Z_1 of Q is singular and codimension $n-1$.
How do we get it to satisfy $(\star)$?

Do the following process:

- Blow up Q along Z_d,
- Blow up the result along (the proper transform of) Z_{d-1},
- ...
- Blow up the result along (the proper transform of) Z_2,
- Blow up the result along (the proper transform of) Z_1.