Clifford Algebras, Division Algebras, and Vector Fields on Spheres

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Introduction

In this talk we will:

- Review the division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$,
- Define Clifford algebras
- Classify all the Clifford modules of a certain type

Then use our knowledge about Clifford modules to yields results about:

- Vector fields on spheres
- Division algebras
- A few other things

Along the way we will see some "shadows" of Bott periodicity.

Division Algebras

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- For $0 \neq x \in \mathbb{K}$, the left-multiplication map L_x is a linear isomorphism.
- So there exists $y \in \mathbb{K}$ such that $L_x y = 1$.

Division Algebras

Complex Numbers

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Division Algebras

Complex Numbers

The first example (other than $\mathbb{R}$!) is $\mathbb{C}$.

- As an $\mathbb{R}$-vector space, $\mathbb{C} \simeq \mathbb{R}^2$.
- The absolute value on $\mathbb{C}$ is the usual norm on $\mathbb{R}^2$.

Division Algebras

Complex Numbers

We can construct $\mathbb{C}$ by the Cayley-Dickson construction:
$\mathbb{C}$ consists of pairs of real numbers with multiplication

$$(a, b)(c, d) = (ac - d^* b, da + bc^*)$$

where $*$ is the conjugation on $\mathbb{R}$ (i.e., it doesn't do anything).

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From this construction, define a conjugation $(a, b)^* = (a^*, -b)$.
This is just the usual $(a + bi)^* = a - bi$.

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Division Algebras

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$\mathbb{C}$ acts on $\mathbb{C}$ by left multiplication.
Identifying $\mathbb{C}$ with $\mathbb{R}^2$ gives a homomorphism $\mathbb{C} \to M_2(\mathbb{R}) : z \mapsto L_z$.
For z with $\|z\| = 1$, we have $L_z \in O(2)$.

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Consider an odd-dimensional sphere $S^{2n-1} \subset \mathbb{R}^{2n}$.
By identifying $\mathbb{R}^{2n}$ with $\mathbb{R}^2 \oplus \cdots \oplus \mathbb{R}^2$,
we can construct a nonvanishing vector field:

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By identifying $\mathbb{R}^{2n}$ with $\mathbb{R}^2 \oplus \cdots \oplus \mathbb{R}^2$,
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At the point $((x_1, y_1), \ldots, (x_n, y_n))$, put the vector
$(i(x_1, y_1), \ldots, i(x_n, y_n)) = ((-y_1, x_1), \ldots, (-y_n x_n))$.

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Division Algebras
Quaternions

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Around 1830 Legendre pointed out that there are no 3-square identities (and therefore no normed division algebra structure on $\mathbb{R}^3$) by showing that $63 = (1^2 + 1^2 + 1^2)(4^2 + 2^2 + 1^2)$ is not a sum of three squares. Was Hamilton aware of this?

Division Algebras

Quaternions

One way to define the quaternions is as follows:
$\mathbb{H}$ consists of 4-tuples of real numbers $a + bi + cj + dk$, satisfying

$$i^2 = j^2 = k^2 = ijk = -1$$

It follows that $ij = k, jk = i, ki = j$, and i, j, k anticommute.

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If $z = a + bi$ and $w = c + di$, we have

$$z + wj = (a + bi) + (c + di)j = a + bi + cj + dk$$

So $\mathbb{H}$ can also be considered as pairs of complex numbers.

Division Algebras

Quaternions

We can also use the Cayley-Dickson construction to define $\mathbb{H}$:
$\mathbb{H}$ consists of pairs of complex numbers with multiplication

$$(u, v)(x, y) = (ux - y^* v, yu + vx^*)$$

where $*$ is the conjugation on $\mathbb{C}$.

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For $q = a + bi + cj + dk$, define $\|q\| = \sqrt{qq^*} = a^2 + b^2 + c^2 + d^2$.
So the norm on $\mathbb{H}$ agrees with the norm on $\mathbb{R}^4$.
One can check that $\|pq\| = \|p\|\|q\|$.

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It follows that

$$u \cdot v = -\frac{1}{2}(uv + vu) \quad u \times v = \frac{1}{2}(uv - vu)$$

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Quaternions

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So A_q preserves the dot product and cross product. So $A_q \in SO(3)$.

Division Algebras

Quaternions

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A_q is a rotation around the vector u by an angle of 2θ.

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Remarks:

- Note that the unit quaternions look like S^3, and form a Lie group.
- The Lie algebra is just vectors in $\mathbb{R}^3$.
- The Lie-theoretic exponential map is the one defined above.
- There is a lot more to be said about all of this...

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Consider an sphere $S^{4n-1} \subset \mathbb{R}^{4n}$.
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$(i(a_1, b_1, c_1, d_1), \ldots, i(a_n, b_n, c_n, d_n)) =$
$((-b_1, a_1, -d_1, c_1), \ldots, (-b_n, a_n, -d_n, c_n))$. Similarly for j and k.

Division Algebras

Octonions

The next division algebra is the octonions, $\mathbb{O}$.

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The octonions were defined by Graves, a friend of Hamilton's, shortly after Hamilton defined the quaternions. He used them to give an 8-square identity (which had been discovered by Degen in 1818). Cayley independently defined them in 1845, and they are often called Cayley numbers.

The Cayley-Dickson construction can be used to define $\mathbb{O}$:
$\mathbb{O}$ consists of pairs of quaternions with multiplication

$$(p, q)(r, s) = (pr - s^* q, sp + qr^*)$$

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Remarks:

- $\mathbb{O}$ is a real 8-dimensional normed division algebra.
- $\mathbb{O}$ is not associative.

Division Algebras

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Division Algebras

Octonions

$\mathbb{O}$ can be used to describe rotations of $\mathbb{R}^8$.

$\mathbb{O}$ acts on $\mathbb{O}$ by left multiplication.
Identifying $\mathbb{O}$ with $\mathbb{R}^8$ gives a *map* $\mathbb{O} \to M_8(\mathbb{R}) : a \mapsto L_a$.
For a with $\|a\| = 1$, we have $L_a \in O(8)$.

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The map is *not* a homomorphism: $L_a L_b x = a(bx) \neq (ab)x = L_{ab}x$.

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$\mathbb{S}$ has a unit and multiplicative inverses, but has zero divisors.
So $\mathbb{S}$ is not a division algebra, and we can stop applying Cayley-Dickson.

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Fix an orthonormal basis $\{e_i\}$ of $\mathbb{R}^k$.
Then $e_i^2 = -1$, and also

$$-2 = (e_i + e_j)^2 = e_i^2 + e_i e_j + e_j e_i + e_j^2 = -2 + e_i e_j + e_j e_i$$

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Therefore $e_i e_j = -e_j e_i$.

So $\mathcal{C}\ell_k$ is generated by k anticommuting "square roots of 1".
These relations are equivalent to the relations $v^2 = -\|v\|^2$.

These algebras (sometimes referred to as geometric algebras) were defined by Clifford in 1876. We will see that $\mathcal{C}\ell_1 \simeq \mathbb{C}$ and $\mathcal{C}\ell_2 \simeq \mathbb{H}$, and $\mathcal{C}\ell_k$ generalizes those algebras. We can't expect $\mathcal{C}\ell_3 \simeq \mathbb{O}$, since the former is associative but the latter is not.

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There are useful for studying $\mathrm{Spin}(n)$ and defining spinors. One can also construct Clifford bundles on a manifold, bundles of Clifford modules, spinor bundles, and other very fancy things.
Lawson and Michelsohn's Spin Geometry has more about this.

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In this talk we will only be interested in finding out when $\mathbb{R}^n$ admits a $\mathcal{C}\ell_k$-module structure. First we will show some basic properties of $\mathcal{C}\ell_k$ to get a feel for it.

Some general observations:

Clifford Algebras

Basic Properties

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There are zero divisors. For example, in $\mathcal{C}\ell_3$:

$$\begin{aligned}(e_1 e_3 + e_2)(e_2 e_3 + e_1) &= e_1 e_3 e_2 e_3 + e_1 e_3 e_1 + e_2^2 e_3 + e_2 e_1 \\ &= e_1 e_2 + e_3 - e_3 - e_1 e_2 = 0.\end{aligned}$$

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For example, $(e_1)(e_1 e_2) = -e_2$.
However, cancellation always occurs in pairs, so there is a $\mathbb{Z}_2$-grading.

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Multiply two general vectors:

$$\begin{aligned}(u_1e_1 + u_2e_2)(v_1e_1 + v_2e_2) &= u_1v_1e_1^2 + u_1v_2e_1e_2 + u_2v_1e_2e_1 + u_2v_2e_2^2 \\ &= -u \cdot v + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} e_1e_2\end{aligned}$$

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So, for example:

$$u \cdot v = -\frac{1}{2}(uv + vu)$$

In fact, this is true in $\mathcal{C}\ell_k$.

Let V be an $\mathbb{R}$-vector space with a quadratic form Q.
Define $\mathcal{C}\ell(V, Q)$ to be the free algebra on V subject to $v^2 = Q(v)$.
(That is, $\bigotimes V/I$, where I is generated by $v \otimes v - Q(v)$)

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We were looking at $\mathcal{C}\ell_k = \mathcal{C}\ell(\mathbb{R}^k, -I)$.
Another Clifford algebra we will need is $\mathcal{C}\ell'_k = \mathcal{C}\ell(\mathbb{R}^k, I)$.
The basic properties of $\mathcal{C}\ell'_k$ are essentially the same as for $\mathcal{C}\ell_k$.

Clifford Algebras

Universal Property

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Clifford Algebras

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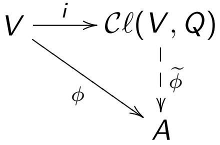
The algebra $\mathcal{C}\ell(V, Q)$ has the following universal property:

Let A be an associative $\mathbb{R}$-algebra.
Define an $\mathbb{R}$-linear map $\phi : V \to A$ such that $\phi(v)^2 = Q(v)1_A$.

Clifford Algebras
Universal Property

The algebra $\mathcal{C}\ell(V, Q)$ has the following universal property:

Let A be an associative $\mathbb{R}$-algebra.
Define an $\mathbb{R}$-linear map $\phi : V \to A$ such that $\phi(v)^2 = Q(v)1_A$.
Then there exists a unique $\widetilde{\phi} : \mathcal{C}\ell(V, Q) \to A$ such that TFDC:



Where $i : V \to \mathcal{C}\ell(V, Q)$ is the natural inclusion.

For this talk we want to know when $\mathbb{R}^n$ has the structure of a $\mathcal{C}\ell_k$-module.

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A homomorphism $\widetilde{\phi} : \mathcal{C}\ell_k \to M_n(\mathbb{R})$ yields such a structure.
Such homomorphisms are induced by maps $\phi : \mathbb{R}^k \to M_n(\mathbb{R})$, such that:

- $\phi(e_i)^2 = -I$
- $\phi(e_i)\phi(e_j) = -\phi(e_j)\phi(e_i)$

That is, we have k matrices satisfying these relations.

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That is, we have k matrices satisfying these relations.

It turns out that determining the (non-)existence of a $\mathcal{C}\ell_k$-module structure will yield useful results about division algebras, vector fields on spheres, n-square identities, cross products, and so on.

Clifford Algebras

Periodicity

Recall that

- $\mathcal{C}\ell_k$ denotes $\mathcal{C}\ell(k, -I)$ (i.e., $v^2 = -\|v\|^2$),
- $\mathcal{C}\ell'_k$ denotes $\mathcal{C}\ell(k, I)$ (i.e., $v^2 = \|v\|^2$)

Let $M_n(\mathbb{K})$ denote $n \times n$ matrices with entries in $\mathbb{K}$.

Theorem

- $\mathcal{C}\ell_1 = \mathbb{C}$
- $\mathcal{C}\ell_2 = \mathbb{H}$

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- $\mathcal{C}\ell_2 = \mathbb{H}$

Proof.

Map $e_1 \mapsto i$.

Map $e_1 \mapsto i$ and $e_2 \mapsto j$ (and $e_1 e_2 \mapsto k$). □

Clifford Algebras

Periodicity

Theorem

- $\mathcal{C}\ell'_1 = \mathbb{R} \oplus \mathbb{R}$
- $\mathcal{C}\ell'_2 = M_2(\mathbb{R})$

Clifford Algebras

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Proof.

Map $e_1 \mapsto (1, -1)$.

Map $e_1 \mapsto \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ and $e_2 \mapsto \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$ $\left(\text{and } e_1 e_2 \mapsto \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\right)$ □

Clifford Algebras

Periodicity

Theorem

(2) $\mathcal{C}\ell_{k+2} \simeq \mathcal{C}\ell'_k \otimes \mathbb{H}$

(2') $\mathcal{C}\ell'_{k+2} \simeq \mathcal{C}\ell_k \otimes M_2(\mathbb{R})$

Clifford Algebras

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Proof of (2).

Let $\{v_i\}$ be a basis for $\mathbb{R}^{k+2}$,
Let $\{e'_i\}$ be the generators for $\mathcal{C}\ell'_k$, and $\{e_1, e_2\}$ for $\mathcal{C}\ell_2$.

Clifford Algebras

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Let $\{e'_i\}$ be the generators for $\mathcal{C}\ell'_k$, and $\{e_1, e_2\}$ for $\mathcal{C}\ell_2$.

Define $u : \mathbb{R}^{k+2} \to \mathcal{C}\ell'_k \otimes \mathcal{C}\ell_2$ by:

- $v_i \mapsto 1 \otimes e_i$ for $i = 1, 2$,
- $v_i \mapsto e'_{i-2} \otimes e_1 e_2$ for $i > 2$.

Clifford Algebras

Periodicity

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- $v_i \mapsto 1 \otimes e_i$ for $i = 1, 2$,
- $v_i \mapsto e'_{i-2} \otimes e_1 e_2$ for $i > 2$.

Check $u(v_i)^2 = -1$ and $u(v_i)u(v_j) + u(v_j)u(v_i) = 0$.
This induces $\widetilde{u} : \mathcal{C}\ell_{k+2} \to \mathcal{C}\ell'_k \otimes \mathcal{C}\ell_2 \simeq \mathcal{C}\ell'_k \otimes \mathbb{H}$.
$\widetilde{u}$ is a bijection (consider it as a map between $\mathbb{R}$-vector spaces). □

Clifford Algebras

Periodicity

Theorem

(4) $\mathcal{C}\ell_{k+4} \simeq \mathcal{C}\ell_k \otimes M_2(\mathbb{H})$

(4') $\mathcal{C}\ell'_{k+4} \simeq \mathcal{C}\ell'_k \otimes M_2(\mathbb{H})$

Clifford Algebras

Periodicity

Theorem

(4) $\mathcal{C}\ell_{k+4} \simeq \mathcal{C}\ell_k \otimes M_2(\mathbb{H})$

(4') $\mathcal{C}\ell'_{k+4} \simeq \mathcal{C}\ell'_k \otimes M_2(\mathbb{H})$

Proof of (4).

Use the previous theorem twice:

$$\begin{aligned}
\mathcal{C}\ell_{k+4} = \mathcal{C}\ell_{(k+2)+2} &\simeq \mathcal{C}\ell'_{k+2} \otimes \mathbb{H} \\
&\simeq (\mathcal{C}\ell_k \otimes M_2(\mathbb{R})) \otimes \mathbb{H} \\
&\simeq \mathcal{C}\ell_k \otimes (M_2(\mathbb{R}) \otimes \mathbb{H}) \simeq \mathcal{C}\ell_k \otimes M_2(\mathbb{H})
\end{aligned}$$

□

Clifford Algebras

Periodicity

Theorem

(8) $\mathcal{C}\ell_{k+8} \simeq \mathcal{C}\ell_k \otimes M_{16}(\mathbb{R})$

(8') $\mathcal{C}\ell'_{k+8} \simeq \mathcal{C}\ell'_k \otimes M_{16}(\mathbb{R})$

Clifford Algebras

Periodicity

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Proof of (8).

Use the previous theorem twice:

$$\begin{aligned}\mathcal{C}\ell_{k+8} = \mathcal{C}\ell_{(k+4)+4} &\simeq \mathcal{C}\ell_{k+4} \otimes M_2(\mathbb{H}) \\ &\simeq (\mathcal{C}\ell_k \otimes M_2(\mathbb{H})) \otimes M_2(\mathbb{H}) \\ &\simeq \mathcal{C}\ell_k \otimes (M_2(\mathbb{H}) \otimes M_2(\mathbb{H})) \simeq \mathcal{C}\ell_k \otimes M_{16}(\mathbb{R})\end{aligned}$$

(Because $\mathbb{H} \otimes \mathbb{H} \simeq M_4(\mathbb{R})$) □

Clifford Algebras

Periodicity

Finally, we can find all of the $\mathcal{C}\ell_k$ (and all of the $\mathcal{C}\ell'_k$):

k	$\mathcal{C}\ell_k$
1	$\mathbb{C}$
2	$\mathbb{H}$
3	
4	
5	
6	
7	
8	

- $\mathcal{C}\ell_3 \simeq \mathcal{C}\ell'_1 \otimes \mathbb{H} \simeq (\mathbb{R} \oplus \mathbb{R}) \otimes \mathbb{H}$.

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- $\mathcal{C}\ell_3 \simeq \mathcal{C}\ell'_1 \otimes \mathbb{H} \simeq (\mathbb{R} \oplus \mathbb{R}) \otimes \mathbb{H}$.
- $\mathcal{C}\ell_4 \simeq \mathcal{C}\ell'_2 \otimes \mathbb{H} \simeq M_2(\mathbb{R}) \otimes \mathbb{H}$.

Clifford Algebras
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3	$\mathbb{H} \oplus \mathbb{H}$
4	$M_2(\mathbb{H})$
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6	
7	
8	

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Clifford Algebras
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6	
7	
8	

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- $\mathcal{C}\ell_4 \simeq \mathcal{C}\ell'_2 \otimes \mathbb{H} \simeq M_2(\mathbb{R}) \otimes \mathbb{H}$.
- $\mathcal{C}\ell_5 \simeq \mathcal{C}\ell_1 \otimes M_2(\mathbb{H}) \simeq \mathbb{C} \otimes M_2(\mathbb{H})$
  But $\mathbb{C} \otimes \mathbb{H} \simeq M_2(\mathbb{C})$.

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1	$\mathbb{C}$
2	$\mathbb{H}$
3	$\mathbb{H} \oplus \mathbb{H}$
4	$M_2(\mathbb{H})$
5	$M_4(\mathbb{C})$
6	
7	
8	

- $\mathcal{C}\ell_3 \simeq \mathcal{C}\ell'_1 \otimes \mathbb{H} \simeq (\mathbb{R} \oplus \mathbb{R}) \otimes \mathbb{H}$.
- $\mathcal{C}\ell_4 \simeq \mathcal{C}\ell'_2 \otimes \mathbb{H} \simeq M_2(\mathbb{R}) \otimes \mathbb{H}$.
- $\mathcal{C}\ell_5 \simeq \mathcal{C}\ell_1 \otimes M_2(\mathbb{H}) \simeq \mathbb{C} \otimes M_2(\mathbb{H})$
  But $\mathbb{C} \otimes \mathbb{H} \simeq M_2(\mathbb{C})$.

Clifford Algebras

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1	$\mathbb{C}$
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4	$M_2(\mathbb{H})$
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6	
7	
8	

- $\mathcal{C}\ell_3 \simeq \mathcal{C}\ell'_1 \otimes \mathbb{H} \simeq (\mathbb{R} \oplus \mathbb{R}) \otimes \mathbb{H}$.
- $\mathcal{C}\ell_4 \simeq \mathcal{C}\ell'_2 \otimes \mathbb{H} \simeq M_2(\mathbb{R}) \otimes \mathbb{H}$.
- $\mathcal{C}\ell_5 \simeq \mathcal{C}\ell_1 \otimes M_2(\mathbb{H}) \simeq \mathbb{C} \otimes M_2(\mathbb{H})$
  But $\mathbb{C} \otimes \mathbb{H} \simeq M_2(\mathbb{C})$.
- $\mathcal{C}\ell_6 \simeq \mathcal{C}\ell_2 \otimes M_2(\mathbb{H}) \simeq \mathbb{H} \otimes M_2(\mathbb{H})$
  But $\mathbb{H} \otimes \mathbb{H} \simeq M_4(\mathbb{R})$.

Clifford Algebras

Periodicity

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1	$\mathbb{C}$
2	$\mathbb{H}$
3	$\mathbb{H} \oplus \mathbb{H}$
4	$M_2(\mathbb{H})$
5	$M_4(\mathbb{C})$
6	$M_8(\mathbb{R})$
7	
8	

- $\mathcal{C}\ell_3 \simeq \mathcal{C}\ell'_1 \otimes \mathbb{H} \simeq (\mathbb{R} \oplus \mathbb{R}) \otimes \mathbb{H}$.
- $\mathcal{C}\ell_4 \simeq \mathcal{C}\ell'_2 \otimes \mathbb{H} \simeq M_2(\mathbb{R}) \otimes \mathbb{H}$.
- $\mathcal{C}\ell_5 \simeq \mathcal{C}\ell_1 \otimes M_2(\mathbb{H}) \simeq \mathbb{C} \otimes M_2(\mathbb{H})$
  But $\mathbb{C} \otimes \mathbb{H} \simeq M_2(\mathbb{C})$.
- $\mathcal{C}\ell_6 \simeq \mathcal{C}\ell_2 \otimes M_2(\mathbb{H}) \simeq \mathbb{H} \otimes M_2(\mathbb{H})$
  But $\mathbb{H} \otimes \mathbb{H} \simeq M_4(\mathbb{R})$.

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4	$M_2(\mathbb{H})$
5	$M_4(\mathbb{C})$
6	$M_8(\mathbb{R})$
7	
8	

- $\mathcal{C}\ell_3 \simeq \mathcal{C}\ell'_1 \otimes \mathbb{H} \simeq (\mathbb{R} \oplus \mathbb{R}) \otimes \mathbb{H}$.
- $\mathcal{C}\ell_4 \simeq \mathcal{C}\ell'_2 \otimes \mathbb{H} \simeq M_2(\mathbb{R}) \otimes \mathbb{H}$.
- $\mathcal{C}\ell_5 \simeq \mathcal{C}\ell_1 \otimes M_2(\mathbb{H}) \simeq \mathbb{C} \otimes M_2(\mathbb{H})$
  But $\mathbb{C} \otimes \mathbb{H} \simeq M_2(\mathbb{C})$.
- $\mathcal{C}\ell_6 \simeq \mathcal{C}\ell_2 \otimes M_2(\mathbb{H}) \simeq \mathbb{H} \otimes M_2(\mathbb{H})$
  But $\mathbb{H} \otimes \mathbb{H} \simeq M_4(\mathbb{R})$.
- $\mathcal{C}\ell_7 \simeq \mathcal{C}\ell_3 \otimes M_2(\mathbb{H}) \simeq (\mathbb{H} \oplus \mathbb{H}) \otimes M_2(\mathbb{H})$.

Clifford Algebras
Periodicity

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1	$\mathbb{C}$
2	$\mathbb{H}$
3	$\mathbb{H} \oplus \mathbb{H}$
4	$M_2(\mathbb{H})$
5	$M_4(\mathbb{C})$
6	$M_8(\mathbb{R})$
7	$M_8(\mathbb{R}) \oplus M_8(\mathbb{R})$
8	

- $\mathcal{C}\ell_3 \simeq \mathcal{C}\ell'_1 \otimes \mathbb{H} \simeq (\mathbb{R} \oplus \mathbb{R}) \otimes \mathbb{H}$.
- $\mathcal{C}\ell_4 \simeq \mathcal{C}\ell'_2 \otimes \mathbb{H} \simeq M_2(\mathbb{R}) \otimes \mathbb{H}$.
- $\mathcal{C}\ell_5 \simeq \mathcal{C}\ell_1 \otimes M_2(\mathbb{H}) \simeq \mathbb{C} \otimes M_2(\mathbb{H})$
  But $\mathbb{C} \otimes \mathbb{H} \simeq M_2(\mathbb{C})$.
- $\mathcal{C}\ell_6 \simeq \mathcal{C}\ell_2 \otimes M_2(\mathbb{H}) \simeq \mathbb{H} \otimes M_2(\mathbb{H})$
  But $\mathbb{H} \otimes \mathbb{H} \simeq M_4(\mathbb{R})$.
- $\mathcal{C}\ell_7 \simeq \mathcal{C}\ell_3 \otimes M_2(\mathbb{H}) \simeq (\mathbb{H} \oplus \mathbb{H}) \otimes M_2(\mathbb{H})$.

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2	$\mathbb{H}$
3	$\mathbb{H} \oplus \mathbb{H}$
4	$M_2(\mathbb{H})$
5	$M_4(\mathbb{C})$
6	$M_8(\mathbb{R})$
7	$M_8(\mathbb{R}) \oplus M_8(\mathbb{R})$
8	

- $\mathcal{C}\ell_3 \simeq \mathcal{C}\ell'_1 \otimes \mathbb{H} \simeq (\mathbb{R} \oplus \mathbb{R}) \otimes \mathbb{H}$.
- $\mathcal{C}\ell_4 \simeq \mathcal{C}\ell'_2 \otimes \mathbb{H} \simeq M_2(\mathbb{R}) \otimes \mathbb{H}$.
- $\mathcal{C}\ell_5 \simeq \mathcal{C}\ell_1 \otimes M_2(\mathbb{H}) \simeq \mathbb{C} \otimes M_2(\mathbb{H})$
  But $\mathbb{C} \otimes \mathbb{H} \simeq M_2(\mathbb{C})$.
- $\mathcal{C}\ell_6 \simeq \mathcal{C}\ell_2 \otimes M_2(\mathbb{H}) \simeq \mathbb{H} \otimes M_2(\mathbb{H})$
  But $\mathbb{H} \otimes \mathbb{H} \simeq M_4(\mathbb{R})$.
- $\mathcal{C}\ell_7 \simeq \mathcal{C}\ell_3 \otimes M_2(\mathbb{H}) \simeq (\mathbb{H} \oplus \mathbb{H}) \otimes M_2(\mathbb{H})$.
- $\mathcal{C}\ell_8 \simeq \mathcal{C}\ell_4 \otimes M_2(\mathbb{H}) \simeq M_2(\mathbb{H}) \otimes M_2(\mathbb{H})$.

Clifford Algebras

Periodicity

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1	$\mathbb{C}$
2	$\mathbb{H}$
3	$\mathbb{H} \oplus \mathbb{H}$
4	$M_2(\mathbb{H})$
5	$M_4(\mathbb{C})$
6	$M_8(\mathbb{R})$
7	$M_8(\mathbb{R}) \oplus M_8(\mathbb{R})$
8	$M_{16}(\mathbb{R})$

- $\mathcal{C}\ell_3 \simeq \mathcal{C}\ell'_1 \otimes \mathbb{H} \simeq (\mathbb{R} \oplus \mathbb{R}) \otimes \mathbb{H}$.
- $\mathcal{C}\ell_4 \simeq \mathcal{C}\ell'_2 \otimes \mathbb{H} \simeq M_2(\mathbb{R}) \otimes \mathbb{H}$.
- $\mathcal{C}\ell_5 \simeq \mathcal{C}\ell_1 \otimes M_2(\mathbb{H}) \simeq \mathbb{C} \otimes M_2(\mathbb{H})$
  But $\mathbb{C} \otimes \mathbb{H} \simeq M_2(\mathbb{C})$.
- $\mathcal{C}\ell_6 \simeq \mathcal{C}\ell_2 \otimes M_2(\mathbb{H}) \simeq \mathbb{H} \otimes M_2(\mathbb{H})$
  But $\mathbb{H} \otimes \mathbb{H} \simeq M_4(\mathbb{R})$.
- $\mathcal{C}\ell_7 \simeq \mathcal{C}\ell_3 \otimes M_2(\mathbb{H}) \simeq (\mathbb{H} \oplus \mathbb{H}) \otimes M_2(\mathbb{H})$.
- $\mathcal{C}\ell_8 \simeq \mathcal{C}\ell_4 \otimes M_2(\mathbb{H}) \simeq M_2(\mathbb{H}) \otimes M_2(\mathbb{H})$.

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2	$\mathbb{H}$
3	$\mathbb{H} \oplus \mathbb{H}$
4	$M_2(\mathbb{H})$
5	$M_4(\mathbb{C})$
6	$M_8(\mathbb{R})$
7	$M_8(\mathbb{R}) \oplus M_8(\mathbb{R})$
8	$M_{16}(\mathbb{R})$

k	$\mathcal{C}\ell_k$
9	$M_{16}(\mathbb{C})$
10	$M_{16}(\mathbb{H})$
11	$M_{16}(\mathbb{H}) \oplus M_{16}(\mathbb{H})$
12	$M_{32}(\mathbb{H})$
13	$M_{64}(\mathbb{C})$
14	$M_{128}(\mathbb{R})$
15	$M_{128}(\mathbb{R}) \oplus M_{128}(\mathbb{R})$
16	$M_{256}(\mathbb{R})$

It will turn out to be useful to know when $\mathbb{R}^n$ has a $\mathcal{C}\ell_k$-module structure.

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Given our classification, this theorem tells us everything we need:

Theorem

If $\mathbb{K}$ is a division algebra,

- *$M_n(\mathbb{K})$ has a unique simple module $\mathbb{K}^n$,*
- *$M_n(\mathbb{K}) \oplus M_n(\mathbb{K})$ has two, inherited from each summand,*
- *Every other module is a direct sum of these.*

(See e.g. Lang's Algebra, chapter XVII)

Let n_k denote the smallest n for which $\mathbb{R}^n$ is a simple $\mathcal{C}\ell_k$-module.

Clifford Algebras

Clifford Modules

Let n_k denote the smallest n for which $\mathbb{R}^n$ is a simple $\mathcal{C}\ell_k$-module.

k	n_k	$\mathcal{C}\ell_k$
1	2	$\mathbb{C}$
2	4	$\mathbb{H}$
3	4	$\mathbb{H} \oplus \mathbb{H}$
4	8	$M_2(\mathbb{H})$
5	8	$M_4(\mathbb{C})$
6	8	$M_8(\mathbb{R})$
7	8	$M_8(\mathbb{R}) \oplus M_8(\mathbb{R})$
8	16	$M_{16}(\mathbb{R})$

k	n_k	$\mathcal{C}\ell_k$
9	32	$M_{16}(\mathbb{C})$
10	64	$M_{16}(\mathbb{H})$
11	64	$M_{16}(\mathbb{H}) \oplus M_{16}(\mathbb{H})$
12	128	$M_{32}(\mathbb{H})$
13	128	$M_{64}(\mathbb{C})$
14	128	$M_{128}(\mathbb{R})$
15	128	$M_{128}(\mathbb{R}) \oplus M_{128}(\mathbb{R})$
16	256	$M_{256}(\mathbb{R})$

Observe that $n_{k+8} = 16n_k$, and that n_k gets a lot bigger than k.

Vector Fields on Spheres

Examples

Problem

For a given n, what is the maximal k such that there exist vector fields $V_1, \ldots, V_k$ on S^{n-1} which are orthonormal at each point?

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We saw earlier that we could use division algebras to construct *some* vector fields. This won't get us very far (as we will see later).

Vector Fields on Spheres

Examples

We can take care of half of the spheres right away:

Theorem (Hairy Ball Theorem)

Even-dimensional spheres have no non-vanishing vector fields.

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Let v be a non-vanishing vector field on S^n. Normalize it.

Define $F_\theta(x) = x\cos\theta + v_x \sin\theta$.
F_θ is a homotopy between the identity 1_S and the antipodal map α.

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Define $F_\theta(x) = x\cos\theta + v_x \sin\theta$.
F_θ is a homotopy between the identity 1_S and the antipodal map α.

So $\deg\alpha = \deg 1_S = 1$. But for even n, $\deg\alpha = -1$, since α is the composition of $n+1$ reflections. □

Vector Fields on Spheres

Application of Clifford Modules

In general, we can get a *lower bound* by studying Clifford modules:

Vector Fields on Spheres

Application of Clifford Modules

In general, we can get a *lower bound* by studying Clifford modules:

Theorem

If $\mathbb{R}^n$ admits the structure of a $\mathcal{C}\ell_k$-module,
then we can construct k orthonormal vector fields on S^{n-1}.

Vector Fields on Spheres

Application of Clifford Modules

In general, we can get a *lower bound* by studying Clifford modules:

Theorem

If $\mathbb{R}^n$ admits the structure of a $\mathcal{C}\ell_k$-module, then we can construct k orthonormal vector fields on S^{n-1}.

Remarks:

- Fixing a basis for $\mathbb{R}^n$, a $\mathcal{C}\ell_k$-module structure on $\mathbb{R}^n$ is a ring homomorphism $\phi : \mathcal{C}\ell_k \to M_n(\mathbb{R})$.
- We can think of this as a choice of matrices $U_i = \phi(e_i)$ satisfying:

$$U_i^2 = -I \quad \text{and} \quad U_i U_j = -U_j U_i \text{ for } i \neq j$$

Vector Fields on Spheres

Application of Clifford Modules

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Computing $\langle U_i x, U_i y \rangle$ just permutes the sum defining $\langle x, y \rangle$.
Choose an orthonormal basis for $\langle , \rangle$.

Vector Fields on Spheres

Application of Clifford Modules

For $x \in S^{n-1}$, we may consider $x \in \mathbb{R}^n$ with $\|x\| = 1$.
The vectors $\{x, U_1x, \ldots, U_kx\}$ are mutually orthogonal:

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Proof Part 1

$$\begin{aligned}
\langle U_i x, x \rangle &= \langle U_i^2 x, U_i x \rangle \quad (\text{since } U_i \in O(n)) \\
&= \langle -x, U_i x \rangle \quad (\text{since } U_i^2 = -1) \\
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$$\begin{aligned} \langle U_i x, U_j x\rangle &= \langle -x, U_i U_j x\rangle \quad (\text{since } U_i \in O(n) \text{ and } U_i^2 = -1) \\ &= \langle x, U_j U_i x\rangle \quad (\text{since } U_i U_j = -U_j U_i) \\ &= \langle U_j x, -U_i x\rangle \quad (\text{since } U_j \in O(n) \text{ and } U_j^2 = -1) \\ &= -\langle U_i x, U_j x\rangle \end{aligned}$$

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So we have k vector fields on S^{n-1}.

Precisely how many vector fields can we construct in this way?

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In other words, given n, what is the largest k such that $\mathbb{R}^n$ admits the structure of a $\mathcal{C}\ell_k$-module?

Vector Fields on Spheres

Application of Clifford Modules

Recall that n_k is the smallest n for which $\mathbb{R}^n$ is a simple $\mathcal{C}\ell_k$-module.

k	n_k	$\mathcal{C}\ell_k$
1	2	$\mathbb{C}$
2	4	$\mathbb{H}$
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$\rho(n)$ is the largest k such that $n_k \mid n$. These are called Radon-Hurwitz numbers.

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Examples:

- For odd n, $\mathbb{R}^n$ is *not* a $\mathcal{C}\ell_k$-module.
- $\mathbb{R}^4$ admits a $\mathcal{C}\ell_3$-module structure.
  $\mathbb{R}^8$ also does (since it is $\mathbb{R}^4 \oplus \mathbb{R}^4$)
  But $\mathbb{R}^8$ is also a $\mathcal{C}\ell_7$-module.

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Observe that:

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So $\rho(n) = 8a + 2^b - 1$.

Vector Fields on Spheres

The Result

Here is a table that shows how many orthonormal VFs we can construct:

	S^1	S^3	S^5	S^7	S^9	S^{11}	S^{13}	S^{15}
# VFs	1	3	1	7	1	3	1	8
	S^{17}	S^{19}	S^{21}	S^{23}	S^{25}	S^{27}	S^{29}	S^{31}
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It turns out that we constructed everything:

- $\rho(n)$ is the *maximal number* of linearly independent VFs on S^{n-1}.
- This is much harder (proved by J.F. Adams in 1962 using Adams operations in K-theory).

Division Algebras

Examples

Problem

When can we make $\mathbb{R}^n$ into a division algebra?

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Remarks:

- There are various theorems with differing assumptions about whether the algebra is commutative, associative, or normed.
- We will *not* require commutativity or associativity.
  We *will* require the algebra to be normed.
- Examples include $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ (that's all of them, in fact).

Division Algebras

Application of Clifford Modules

Let's see what Clifford modules say about such division algebras:

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Theorem

Let $\mathbb{K}$ be a finite-dimensional normed division algebra over $\mathbb{R}$.
If $\dim \mathbb{K} = n \ (> 1)$, then $\mathbb{R}^n$ admits the structure of a $\mathcal{C}\ell_{n-1}$-module.

We will show that this implies $n = 2$, 4, or 8.

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Remarks:

- Corollary: $\dim \mathbb{K} = n \implies S^{n-1}$ is parallelizable.
  That isn't useful unless we know which spheres are parallelizable.
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- Corollary: $\dim \mathbb{K} = n \implies S^{n-1}$ is parallelizable.
  That isn't useful unless we know which spheres are parallelizable.
  (our lower bound $\rho(n)$ is sharp, but that is hard to prove).
- If we drop the normed condition, it still implies S^{n-1} is parallelizable.
  It *no longer* implies $\mathbb{R}^n$ is a $\mathcal{C}\ell_{n-1}$-module.

Division Algebras

Application of Clifford Modules

Let $\mathbb{K}$ be our division algebra, $\dim \mathbb{K} = n$.

As an $\mathbb{R}$-vector space, it is $\mathbb{R}^n$, and it has some multplication.

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We will *assume* the norm comes from an inner product $\langle , \rangle$ on $\mathbb{R}^n$.
(This isn't obvious, but it can be proven)
Fix an orthonormal basis for $\langle , \rangle$.

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Let $\mathrm{Im}(\mathbb{K})$ denote the elements of $\mathbb{K}$ orthogonal to 1 ($\mathbb{R}^{n-1}$ as a VS).
Left multiplication is an $\mathbb{R}$-linear map $\mathbb{K} \to \mathbb{K}$. So there is a map

$$\mathrm{Im}(\mathbb{K}) \xrightarrow{\phi} M_n(\mathbb{R}) : v \mapsto \left(x \xmapsto{L_v} vx \right)$$

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The strategy is to show that for $v \in \mathrm{Im}(\mathbb{K})$ with $\|v\| = 1$, $L_v^2 = -I$.
That will induce a map $\widetilde{\phi} : \mathcal{C}\ell_{n-1} \to M_n(\mathbb{R})$.

Division Algebras

Application of Clifford Modules

Let $v \in \mathrm{Im}(\mathbb{K})$, $\|v\| = 1$.
$L_v \in O(n)$ (since $\|L_v x\| = \|vx\| = \|v\|\|x\| = \|x\|$).

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Proof that $L_v^2 = -I$

Let $w = (v + 1)/\sqrt{2}$.
Since $\|w\| = 1$, $L_w = (L_v + I)/\sqrt{2} \in O(n)$.

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Since $\|w\| = 1$, $L_w = (L_v + I)/\sqrt{2} \in O(n)$.

Therefore:

$$\begin{aligned} I = L_w L_w^* &= \frac{1}{2}(L_v + I)(L_v^* + I) \\ &= \frac{1}{2}(L_v L_v^* + L_v + L_v^* + I) = I + \frac{1}{2}(L_v + L_v^*) \end{aligned}$$

So $L_v^2 = (-L_v^*)(L_v) = -I$.

Division Algebras

The Result

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This only happens for $n = 2$, 4, or 8. After $n = 8$, $n_{n-1} > n$.

We proved:

Theorem

Let $\mathbb{K}$ *be a finite-dimensional normed division algebra over* $\mathbb{R}$*.*
Then $\dim \mathbb{K} = 1, 2, 4,$ *or* 8*.*

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Theorem

Let $\mathbb{K}$ be a finite-dimensional normed division algebra over $\mathbb{R}$. Then $\dim \mathbb{K} = 1, 2, 4,$ or 8.

Remarks:

- The same result is true if we drop the normed condition.
- This is much harder (proved by Kervaire, and, independently, by Bott and Milnor, in 1958).
- It's *not* much harder to show $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, or $\mathbb{O}$ (Hurwitz's theorem). This part is *not* true if we drop the normed condition.

Applications

Square Identities

Consider this two-square identity:

$$\begin{aligned}(a_1^2 + a_2^2)(b_1^2 + b_2^2) &= c_1^2 + c_2^2 \\ c_1 &= a_1 b_1 - a_2 b_2 \\ c_2 &= a_1 b_2 + a_2 b_1\end{aligned}$$

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There is also Euler's four-square identity:

$$\begin{aligned}(a_1^2 + a_2^2 + a_3^2 + a_4^2)(b_1^2 + b_2^2 + b_3^2 + b_4^2) &= c_1^2 + c_2^2 + c_3^2 + c_4^2 \\ c_1 &= a_1 b_1 - a_2 b_2 - a_3 b_3 - a_4 b_4 \\ c_2 &= a_1 b_2 + a_2 b_1 + a_3 b_4 - a_4 b_3 \\ c_3 &= a_1 b_3 - a_2 b_4 + a_3 b_1 + a_4 b_2 \\ c_4 &= a_1 b_4 + a_2 b_3 - a_3 b_2 + a_4 b_1\end{aligned}$$

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There is also Degen's eight-square identity (I'm not typing that one!)

Applications

Square Identities

These identities just come from the normed division algebra structures on $\mathbb{R}^n$ for $n = 2, 4, 8$: $\|a\|^2 \|b\|^2 = \|c\|^2$, $c = ab$, where $a, b, c \in \mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$.

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Define an n-square identity to be an expression of the form $\left(\sum a_i^2\right)\left(\sum b_i^2\right) = \sum c_i^2$, where c_i is bilinear in the a's and b's.

There only exist 1-, 2-, 4-, and 8-square identities.

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Start by defining an $\mathbb{R}$-algebra structure on $\mathbb{R}^n$ by:

$$(a_1, \ldots, a_n)(b_1, \ldots, b_n) = (c_1, \ldots, c_n)$$

This algebra preserves the norm on $\mathbb{R}^n$. It has no zero-divisors.

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To fix, this we perform a "mutation":

- Choose $u \in \mathbb{R}^n$ with $\|u\| = 1$.
- Define a new product $a * b = (R_u^{-1} a)(L_u^{-1} b)$,
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The new product still gives us an $\mathbb{R}$-algebra preserving $\| \cdot \|$.
The multiplicative identity is u^2.

We can't apply our division algebra theorem; there is no apparent unit!

To fix, this we perform a "mutation":

- Choose $u \in \mathbb{R}^n$ with $\|u\| = 1$.
- Define a new product $a * b = (R_u^{-1} a)(L_u^{-1} b)$,
  (where L_u and R_u are left and right multiplication by u)

The new product still gives us an $\mathbb{R}$-algebra preserving $\| \cdot \|$.
The multiplicative identity is u^2.

Let $x \in \mathbb{R}^n$ and $y = L_u^{-1} x$. Then:

$$u^2 * x = u^2 * L_u y = (R_u^{-1} u^2)(L_u^{-1} L_u y) = uy = x$$

(Where $R_u^{-1} u^2 = u$ since $R_u u = u^2$).

Applications

Square Identities

This mutation turned $\mathbb{R}^n$ into finite-dimensional normed division algebra over $\mathbb{R}$, so $n = 1, 2, 4$, or 8, proving the theorem.

Applications

Cross Products

Cross products exist in $\mathbb{R}^3$ and $\mathbb{R}^7$ ($\mathrm{Im}(\mathbb{H})$ and $\mathrm{Im}(\mathbb{O})$, respectively).

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Theorem

Suppose there is a cross product on $\mathbb{R}^n$, $n \geq 3$, such that

- *$u \times v$ is bilinear in u and v,*
- *$u \times v$ is perpendicular to u and v,*
- $\|u \times v\|^2 = \|u\|^2 \|v\|^2 - (u \cdot v)^2$

Then $n = 3$ or 7.

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Existence of such a cross product makes $\mathbb{R} \oplus \mathbb{R}^n$ into a normed division algebra by defining:

$$(a, v)(b, w) = (ab - v \cdot w, aw + bv + v \times w)$$

(This from Massey's paper "Cross products of vectors in higher dimensional Euclidean spaces")

Bott Periodicity

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But in fact we can drop the "normed". That turns out to be much harder. (Kervaire, Bott, Milnor 1958).

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Theorem

There are at least $\rho(n)$ *linearly independent vector fields on* S^{n-1}.

But in fact there are also *at most* $\rho(n)$. Also much harder. (Adams 1962).

Bott Periodicity

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The statement of (one version of) Bott periodicity:

$$\begin{aligned}
\widetilde{KO}(S^1) &\simeq \mathbb{Z}_2 \\
\widetilde{KO}(S^2) &\simeq \mathbb{Z}_2 \\
\widetilde{KO}(S^3) &\simeq 0 \\
\widetilde{KO}(S^4) &\simeq \mathbb{Z} \\
\widetilde{KO}(S^5) &\simeq 0 \\
\widetilde{KO}(S^6) &\simeq 0 \\
\widetilde{KO}(S^7) &\simeq 0 \\
\widetilde{KO}(S^8) &\simeq \mathbb{Z}
\end{aligned}$$

And $\widetilde{KO}(S^{8+k}) \simeq \widetilde{KO}(S^k)$.

Bott Periodicity

There are isomorphisms $L_k \simeq \widetilde{KO}(S^k)$,
where $L_k \simeq \mathrm{coker}(N_k \to N_{k-1})$,
where N_k is the free abelian group generated by simple $\mathcal{C}\ell_k$-modules, and $N_k \to N_{k-1}$ is induced by an inclusion $\mathcal{C}\ell_{k_1} \to \mathcal{C}\ell_k$.

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