Curvature forms and characteristic classes

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Grassmannians

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The infinite Grassmannian $G_n(\mathbb{R}^\infty)$ is the direct limit of the sequence:

$$G_n(\mathbb{R}^n) \subset G_n(\mathbb{R}^{n+1}) \subset \cdots \subset G_n(\mathbb{R}^{n+k}) \subset \cdots$$

(Direct limit: Take $\bigcup_{k \geq 0} G_n(\mathbb{R}^{n+k})$ and choose the finest topology such that every inclusion is continuous)

Grassmannians

The tautological bundle

We can form an n-plane bundle over $G_n(\mathbb{R}^{n+k})$:
Take $X \in G_n(\mathbb{R}^{n+k})$. The fiber over X is X itself.
So the total space consists of pairs (n-plane in $\mathbb{R}^{n+k}$, vector in that plane).

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- $G_1(\mathbb{R}^2) \simeq S^1$, and the tautological bundle "is" the Möbius strip.

Grassmannians

Generalized Gauss Map

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In fact, this works for any n-plane bundle over M, for large enough k:

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For any n-plane bundle ξ over M, $\xi = g^*(\gamma^n)$ for some $g : M \to G_n(\mathbb{R}^\infty)$.

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Bundles over M are isomorphic iff their classifying maps are homotopic.
Note: M must be paracompact (open covers admit locally finite refinements). This includes manifolds, metric spaces, CW complexes...

Classifying Spaces

The structure group

Definition

- A G-**atlas** is a local trivialization of a bundle with transition functions $U \cap V \to G$.
- A G-**bundle** is a vector bundle with a G-atlas.
- G is called the **structure group**.

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Let M be n-dimensional.

- TM is a $GL(n, \mathbb{R})$-bundle.
- If M has a metric, TM is an $O(n)$-bundle.
- If M is also orientable, TM is an $SO(n)$-bundle.
- If M has an almost complex structure, TM is a $U(n)$-bundle.

Classifying Spaces

The functors b_G and B

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- $BO(n) = G_n(\mathbb{R}^\infty)$.
- $BSO(n) = \widetilde{G}_n(\mathbb{R}^\infty)$ (Grassmannian of oriented planes).
- $BU(n) = G_n(\mathbb{C}^\infty)$.

Characteristic Classes

Cohomology of BG

Now we can study of the twisting of a bundle using the classifying map.
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- $H^*(BO(n), \mathbb{Z}_2) \simeq \mathbb{Z}_2[w_1, \ldots, w_n]$.
- $H^*(BSO(n), \mathbb{Z}_2) \simeq \mathbb{Z}_2[w_2, \ldots, w_n]$.
- $H^*(BU(n), \mathbb{R}) \simeq \mathbb{R}[c_1, \ldots, c_n]$ (or $\mathbb{Z}$).

For a ring R with $1/2$ (eg $\mathbb{R}$),

- $H^*(BSO(2n+1), R) = R[p_1, \ldots, p_n]$
- $H^*(BSO(2n), R) = R[p_1, \ldots, p_{n-1}, e]/(e^2 = p_{n/2})$.

We call w_i Stiefel-Whitney classes, c_i Chern classes, p_i Pontryagin classes, and e the Euler class.

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A **characteristic class** is an assignment c of a class in $h^*(M)$ given a G-bundle over M.
It is natural: For ξ over N and $f : M \to N$, $c(f^*\xi) = f^*c(\xi)$.

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In fact, *all* characteristic classes must be of this form:

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Theorem

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This follows from contravariant Yoneda's lemma: b_G is representable, and h^* can be regarded as a functor to *Set*.

A sampling of results:

- M is orientable iff $w_1(M) = 0$.
- M is the boundary of a compact manifold iff $w_i = 0\ \forall i$.
- If $\mathbb{R}P^{2^r}$ is immersed in $\mathbb{R}^{2^r+k}$, then $k \geq 2^r - 1$.
- If M has a q-frame, then $w_n = \cdots = w_{n-q+1} = 0$.
- Oriented 3-manifolds are parallelizable.

Curvature forms

Vector-valued forms

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Definition

Define ξ-valued k-forms by $\Omega^k(\xi) = \Omega^k(M) \otimes_{\Omega^0(M)} \Omega^0(\xi)$.

Curvature forms

Reinterpreting what we have done

Put a metric $\langle , \rangle$ and connection ∇ on M.
The **connection forms** are defined by $\nabla_X E_i = \omega_i^j(X) E_j$.

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This motivates a definition of a connection on a general vector bundle ξ:

Definition

A connection on ξ is an $\mathbb{R}$-linear map $\nabla : \Omega^0(\xi) \to \Omega^1(\xi)$,
such that $\nabla(fV) = df \otimes V + f \nabla V$.

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Let $\Omega = dA - A \wedge A$, the matrix of **curvature forms**.

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Remarks:

- If ∇ compatible, then A, Ω antisymmetric WRT orthonormal frames.
- There is a natural way to pull back connections.

Invariant polynomials

We want to apply a polynomial P to Ω and get a globally defined form.
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So $\sigma_k(\Omega)$ is a globally defined form on M.

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The Pfaffian

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- $\mathrm{Pf}\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} = a$
- $\mathrm{Pf}\begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix} = af - be + dc$

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One can show $\mathrm{Pf}(BAB^T) = \mathrm{Pf}(A)\det(B)$.
For B orthogonal, this is invariant.
So for an orientable $2n$-plane bundle, $\mathrm{Pf}(\Omega)$ is a globally defined $2n$-form.

Invariant polynomials

It turns out that for invariant P:

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If it is natural, then it has to be a characteristic class.

If ξ is a bundle on N with connection ∇, and $f : M \to N$,
then $f^*(P(\Omega_\nabla)) = P(\Omega_{f^*\nabla})$.
So it is natural.

Invariant polynomials

Characteristic classes

One can show that:

- For a bundle with structure group $SO(2n)$, $\mathrm{Pf}\left(\frac{\Omega}{2\pi}\right) = e$.
- For a complex bundle, $\det\left(I + \frac{t\Omega}{2\pi i}\right) = \sum c_k t^k$.

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For a surface, $\Omega = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}$

So $\mathrm{Pf}\left(\frac{\Omega}{2\pi}\right) = \frac{\omega}{2\pi} = \frac{K}{2\pi} dV$.

Chern-Gauss-Bonnet

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(Very rough) sketch of proof (M compact):

- Let $\pi : E \to M^n$ be a k-plane bundle.
- The Thom isomorphism theorem says $\Phi : H^i(M) \simeq H^{i+k}(T(E))$.
- $\Phi(x) = (\pi^* x) \wedge U$, where U is the Thom class.
- If $s : M \to E$ is a section, $s^* U$ is the Euler class e.
- Choose s for which Poincaré-Hopf applies.
- Then show $\int_M s^* U$ is the sum of indices of zeros of s, which is $\chi(M)$.

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We can write e in terms of Ω to get:

Theorem

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The end!
For details see Spivak I.11, V.13 and Milnor-Stasheff